Effective Fokker-Planck equation for birhythmic modified van der Pol oscillator

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Effective Fokker-Planck equation for birhythmic modified van der Pol oscillator

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We present an explicit solution based on the phase-amplitude approximation of the *Fokker-Planck* equation associated with the *Langevin* equation of the birhythmic modified *van der Pol* system. The solution enables us to derive probability distributions analytically as well as the activation energies associated with switching between the coexisting different attractors that characterize the birhythmic system. Comparing analytical and numerical results we find good agreement when the frequencies of both attractors are equal, while the predictions of the analytic estimates deteriorate when the two frequencies depart. Under the effect of noise, the two states that characterize the birhythmic system can merge, inasmuch as the parameter plane of the birhythmic solutions is found to shrink when the noise intensity increases. The solution of the *Fokker-Planck* equation shows that in the birhythmic region, the two attractors are characterized by very different probabilities of finding the system in such a state. The probability becomes comparable only for a narrow range of the control parameters, thus the two limit cycles have properties in close analogy with the thermodynamic phases. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4766678]

The *van der Pol* oscillator is a model of self-oscillating system that exhibits periodic oscillations. A modified version—essentially a higher order polynomial dissipation—has been proposed as a model equation for enzyme dynamics. This model is very interesting as a paradigm for birhythmicity, it contains multiple stable attractors with different natural frequencies, therefore, it can describe spontaneous switching from one attractor to another under the influence of noise. The noise induced transitions between different attractors depend upon the different stability properties of the attractors and are usually investigated by means of extensive *Langevin* simulations. We show that the associated *Fokker-Planck* equation, in the phase-amplitude approximation, is analytically solvable. The phase amplitude approximation requires a single frequency, and therefore fails when the two frequencies of the birhythmic system are significantly different. However, the approximation is not severe, for it explains the main features of the system when compared to the numerical simulations of the full model. The approximated *Fokker-Planck* equation reveals the underlining structure of an effective potential that separates the different attractors with different frequency, thus explaining the remarkable differences of the stability between the coexisting attractors that give rise to birhythmicity. Moreover, it reveals that the noise can induce the stochastic suppression of the bifurcation that leads to birhythmicity.

Finally, the approximated solution shows that the system is located with overwhelming probability in one attractor, thus being the dominant attractor. Which attractor is dominant depends upon the external control parameters. This is in agreement with the general expectation that in bistable systems the passage from an attractor to the other resembles phase transitions, since only in a very narrow interval of the external parameters it occurs in both directions with comparable probabilities.

I. INTRODUCTION

A stochastic dynamical system is a dynamical system under the effects of noise. Such effects of fluctuations have been of interest for over a century since the celebrated work of *Einstein*.[1] Fluctuations are classically referred to as "noisy" or "stochastic" when their suspected origin implicates the action of a very large number of variables or degrees of freedom. For a linear system, this leads to the phenomenon of diffusion, while the coupling of noise to non-linear deterministic equations can lead to non-trivial effects.[2,3] For example, noise can stabilize unstable equilibria and shift bifurcations, i.e., the parameter value at which the dynamics changes qualitatively.[4,5] Noise can also lead to transitions between coexisting deterministic stable states or attractors such as in birhythmic or bistable system.[6] Moreover, noise can induce new stable states that have no deterministic counterpart, for instance, noise excites internal

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modes of oscillation, and it can even enhance the response of a nonlinear system to external signals.[7,8]

In this paper, we investigate analytically the effects of an additive noise on a special bistable system that displays birhythmicity— coexisting attractors that are characterized by different frequencies.[9–16] We examine a birhythmic self-sustained system described by the modified *van der Pol* oscillator, subjected to an additive Gaussian white noise.[6]

Our main aim is to use the phase-amplitude approximation,[17] a standard technique for van der Pol[17] and van der Pol-like systems,[18] to derive an effective *Fokker-Planck* equation[19] that can be analytically managed. This allows us to analytically derive the activation energies associated to the switching between different attractors.[6,20] The analytical solution of the approximated model is not limited to vanishingly small noise intensity as it was done for the numerical estimate of the escape time[6] to derive the pseudopotential.[21] Another purpose of the present paper is to verify, with numerical simulations, that in spite of the approximations the analytical probability distribution is reliable.

The paper is organized as follows: Section II presents the modified *van der Pol* system with an additive Gaussian white noise. Section III deals with the derivation and analysis of an effective *Fokker-Planck* equation for the birhythmic modified *van der Pol* oscillator. The probability distribution given by the approximated *Fokker-Planck* equation is analyzed and the activation energies are derived. In Sec. IV, we integrate numerically the stochastic second order differential equation and discuss the results. Section V concludes.

II. THE BIRHYTHMIC PROPERTIES OF THE NOISY MODEL

A. The modified van der Pol oscillator with an additive noise

The model considered is a *van der Pol* oscillator with a nonlinear dissipation of higher polynomial order described by the equation (overdots as usual stand for the derivative with respect to time)

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + x = 0. \quad (1)$$

This model was proposed by Kaiser[22] as more appropriate than the *van der Pol* oscillator to describe certain specific processes in biophysical systems. In fact the modified *van der Pol*-like oscillator described by Eq. (1) is used to model coherent oscillations in biological systems, such as an enzymatic substrate reaction with ferroelectric behavior in brain waves models (see Refs. 23–26 for more details). From the standpoint of nonlinear dynamics, it represents a model which exhibits an extremely rich bifurcation behavior. The quantities α and β are positive parameters which measure the degree of tendency of the system to a ferroelectric instability compared to its electric resistance, while μ is the parameter that tunes nonlinearity.[23] The model Eq. (1) is a nonlinear self-sustained oscillator which possesses more than one stable limit-cycle solution.[27] Such systems are of interest especially in biology, for example, to describe the coexistence of two stable oscillatory states, as in enzyme reactions.[28]

Another example is the explanation of the existence of multiple frequency and intensity windows in the reaction of biological systems when they are irradiated with very weak electromagnetic fields[24,27,29–32]. Moreover, the model under consideration offers general aspects concerning the behavior of nonlinear dynamical systems. *Kaiser* and *Eichwald*[32] have analyzed the super-harmonic resonance structure, while *Eichwald* and *Kaiser*[22] have found symmetry-breaking crisis and intermittency.

In Ref. 23, an analytical approximation has been derived for the coexisting oscillations of the two attractors with different natural frequencies for the deterministic part of the model equation. A numerical investigation of the escape times (and hence of the activation energies) has suggested that the stability properties of the attractors can be very different.[6] It has further been shown that time delayed feedback leads to stabilization,[33] also in the presence of external noise.[20]

Noise can enter the system, for instance, through the electric field applied to the excited enzymes which depends on the external chemical influences or through the flow of enzyme molecules. One can therefore assume that the environmental influence contains a random perturbation and to postulate that the activated enzymes are subject to a random excitation governed by the *Langevin* version of Eq. (1), namely:

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + x = \Gamma(t). \quad (2)$$

$\Gamma(t)$ can be assumed to be an additive Gaussian white noise with arbitrary amplitude D[17] and it has the properties

$$\begin{aligned} \langle \Gamma(t) \rangle &= 0, \\ \langle \Gamma(t), \Gamma(t') \rangle &= 2D\delta(t - t'), \end{aligned} \quad (3)$$

which completely determine its statistical features. The noise term is here treated as external,[36] i.e., due to a disturbance from the environment and not subject to the fluctuation dissipation theorem.

B. Birhythmic properties

Without noise ($\Gamma = 0$), Eq. (2) reduces to the modified version of the *van der Pol* oscillator (1) which has steady-state solutions that depend on the parameters α, β, and μ and correspond to attractors in state space. The dynamical attractors of the free-noise modified *van der Pol* Eq. (1) have been determined analytically, the expressions of the amplitudes A_i and frequency Ω_i ($i = 1, 2, 3$) of the limit-cycle solutions have been established in Refs. 6, 20, and 23, in which the periodic solutions of the modified *van der Pol* oscillator (1) are approximated by

$$x(t) = A\cos\Omega t. \quad (4)$$

The amplitude A is independent of the coefficient μ up to corrections of the order μ^2 and implicitly given by the relation

$$\frac{5\beta}{64}A^6 - \frac{\alpha}{8}A^4 + \frac{1}{4}A^2 - 1 = 0. \quad (5)$$

The coefficient μ enters in the expression for the frequency Ω as a second order correction

$$\Omega = 1 + \mu^2\omega_2 + o(\mu^3), \tag{6}$$

thus the deviations of the frequency from the linear harmonic solution are characterized by an amplitude dependent frequency[26]

$$\omega_2 = \frac{93\beta^2}{65536}A^{12} - \frac{69\alpha\beta}{16384}A^{10} + \left(\frac{67\beta}{8192} + \frac{3\alpha^2}{1024}\right)A^8 \\ - \left(\frac{73\beta}{2048} + \frac{\alpha}{96}\right)A^6 + \left(\frac{1}{128} + \frac{\alpha}{24}\right)A^4 - \frac{3}{64}A^2. \tag{7}$$

Depending on the values of the parameters α and β, the modified *van der Pol* oscillator possesses one or three limit cycles. In fact, Eq. (5) can give rise to one or three positive real roots that correspond to one stable limit cycle or three limit cycle solutions (of which two are stable and one is unstable), respectively. The dynamical attractors and bir-hythmicity (i.e., the coexistence between two stable regimes of limit cycle oscillations) are numerically found solving the amplitude Eq. (7).[6] The three roots A_1, A_2, and A_3 denote the inner stable orbit, the unstable orbit, and the outer stable orbit, respectively. When three limit cycles are obtained, Eq. (6) supplies the frequencies $\Omega_{1,2,3}$ in corre-spondence of the roots $A_{1,2,3}$. Being one of the attractors unstable, the system only displays two frequencies $\Omega_{1,3}$ (and hence birhythmicity) at two different amplitudes $A_{1,3}$, while the unstable limit cycle of amplitude A_2 represents the separatrix between the basins of attraction of the stable limit cycles. We show in Fig. 1 the region of existence of birhythmicity in the two parameters phase space $(\alpha$-$\beta)$[23,26] (the two coexisting stable limit cycle attractors can be found in Ref. 6). The question we want to address is the influence of noise on the above properties investigating the response of an additive Gaussian white noise in the phase-amplitude limit. In Ref. 6, the system has been numerically tackled in the regime of vanishingly small noise. In this limit, the escape rate gives an effective potential that acts as an activation barrier. We employ the phase-amplitude approximation that should be both faster (being analytical)
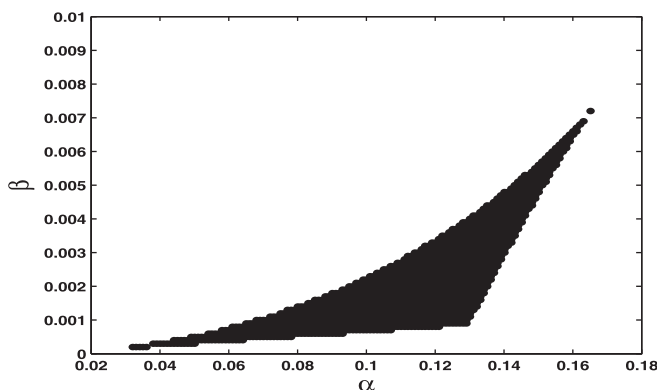

FIG. 1. Parameters domain for the existence of a single limit cycle (white area) and three limit cycles (black area) solutions of Eq. (1) for $\mu = 0.1$.

and more accurate at finite values of the noise, as will be discussed in Sec. III.

III. ANALYTICAL ESTIMATES

The analytic results on the deterministic system are based on the approximated cycle given by Eq. (4). Quite naturally, one can treat the noise in the system starting from such approxi-mation. To this extent, we rewrite the *Langevin* Eq. (2) in a sys-tem of two coupled first order differential equations:

$$\dot{x} = u, \\ \dot{u} = \mu\left(1 - x^2 + \alpha x^4 - \beta x^6\right)u - x + \Gamma. \tag{8}$$

We seek for solution in the context of the phase-amplitude approximation, i.e., letting the amplitude and the phase of Eq. (4) to be time dependent[17]

$$x = A(t)\cos(\Omega t + \phi(t)), \\ u = -A(t)\omega_0\sin(\Omega t + \phi(t)). \tag{9}$$

Inserting Eq. (9) into Eq. (8), one retrieves a system of two *Langevin* equations for the amplitude $A(t)$ and phase $\phi(t)$ variables, that is, of course, as difficult to manage as the origi-nal model (8). We will follow the standard analysis of nonlin-ear oscillators[18,34] that consists in assuming that in a period $2\pi/\Omega$ the variables $A(t)$ and $\phi(t)$ do not change significantly, so one can average the effect of the random perturbation.[20] Although in principle this method also relies on the smallness of the noise, since the averaging requires that the approximate solution (9) is not significantly altered in a cycle $2\pi/\Omega$, the procedure has proven very robust in a similar *van der Pol-Duffing* oscillator.[18] It is important to note that for $\mu = 0$ the system reduces to the harmonic oscillator, as described by the solution Eq. (9) with constant amplitude and phase. Since we are interested in the influence of noise D and nonlinear dissi-pation (α and β) in birhythmic systems, we keep the parameter μ small ($\mu = 0.1$). If the present model is employed to model the population of enzyme molecules, the parameter represents the difference between the thermal activated polarization and the external field induced polarization.[27] However, we note that for a birhythmic system a further difficulty occurs: the system has two different frequencies $\Omega_1 \neq \Omega_3$, while the approximation (9) is monorhythmical. Assuming that the two frequencies are not too different, we insert Eq. (9) into Eq. (8) and average, to retrieve the effective (and simpler) *Langevin* equation for the amplitude A and phase ϕ variables:

$$\dot{\phi} = -\frac{\Omega^2 - 1}{2\Omega} - \sqrt{\frac{D}{2A\Omega^2}}\langle\Gamma(t)\rangle, \\ \dot{A} = \frac{\mu A}{2}\left[\left(1 - \frac{1}{4}A^2 + \frac{1}{8}\alpha A^4 - \frac{5}{64}\beta A^6\right)\right] - \sqrt{\frac{D}{2\omega_0^2}}\langle\Gamma(t)\rangle. \tag{10}$$

We thus study the system in the slow averaged variables; for the slow variables the average noise can still be considered white and uncorrelated,[17] and the *Fokker-Planck* equation associated to the *Langevin* model (10) reads

$$\frac{\partial P}{\partial t} = -\frac{\partial S^\phi}{\partial \phi} - \frac{\partial S^A}{\partial A}, \tag{11}$$

where $S = S^\phi + S^A$ is the probability current defined by

$$S^\phi = K_1^\phi P - \frac{\partial}{\partial \phi}(K_2^{\phi,\phi}P),$$
$$S^A = K_1^A P - \frac{\partial}{\partial A}(K_2^{A,A}P). \tag{12}$$

The drift coefficients K_1^ϕ and K_1^A associated to Eq. (10) read as

$$K_1^\phi = -\frac{\Omega^2}{2\Omega},$$
$$K_1^A = \frac{\mu A}{2}\left[1 - \frac{1}{4}A^2 + \frac{1}{8}\alpha A^4 - \frac{5}{64}\beta A^6\right] + \frac{D}{2\Omega^2 A}. \tag{13}$$

The off diagonal diffusion coefficients $K_2^{\phi,A}$ and $K_2^{A,\phi}$ vanish, while the diagonal coefficient reads as

$$K_2^{\phi,\phi} = \frac{D}{(\Omega A)^2},$$
$$K_2^{A,A} = \frac{D}{2\Omega^2}. \tag{14}$$

We seek for stationary solutions, $\partial P/\partial t = 0$ of Eq. (11). We note that in the averaged equation (9) for the phase A the phase ϕ does not appear, and therefore the integration over all phases gives rises to a normalization constant. We therefore only seek solutions for the probability distribution associated with the constant probability current, $S^A = const$. Moreover, since the probability distribution must vanish for $A = \infty$, we can set the constant to 0. Finally, the equation for the radial part of the probability distribution P reads as

$$S^A = 0 \Rightarrow K_1^A P = \frac{d}{dA}(K_2^{A,A}P), \tag{15}$$

or, explicitly

$$P(A) = cA\exp\left\{\frac{\mu\Omega^2}{2D}A^2\left[1 - \frac{1}{8}A^2 + \frac{1}{24}\alpha A^4 - \frac{5}{256}\beta A^6\right]\right\}, \tag{16}$$

where c is a constant of normalization. This solution contains as particular cases the harmonic oscillator ($\mu < 0$, $\alpha = \beta = 0$, and discarding the $A^2/8$ term) and the standard *van der Pol* oscillator ($\mu > 0$, $\alpha = \beta = 0$).

The probability distribution is in general very asymmetric, for most of the parameters α or β one can localize the probability function around a single orbit. Before proceeding further in our analysis, it should be noted that the peaks of the probability distribution can be located using the following equation:

$$\frac{d\log(P)}{dA} = 0 \Rightarrow \left[\frac{5}{64}\beta A^6 - \frac{1}{8}\alpha A^4 + \frac{1}{4}A^2 - 1\right]A^2 - \frac{D}{\mu\Omega^2} = 0. \tag{17}$$

For $D = 0$, the amplitude (17) coincides with the deterministic amplitude equation (5).[20] In Fig. 2, we report the influence of
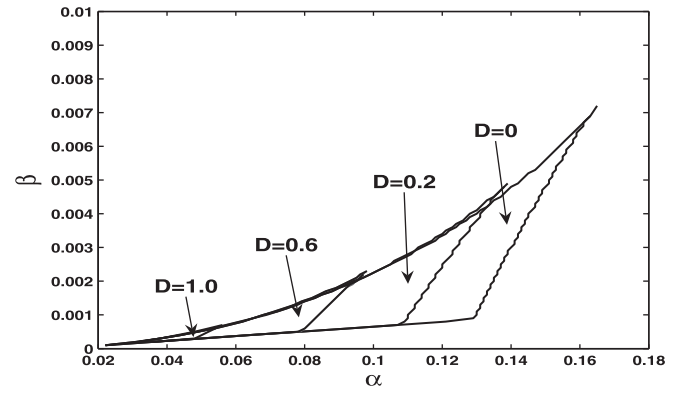

FIG. 2. Effect of the noise intensity D on the boundary between the region of one and three limit-cycle solutions in the parametric (α,β)-plane of the Fokker-Planck Eq. (11)) for $\mu = 0.1$ as in Fig. 1.

the noise intensity D on the region of multi-limit cycle orbits of Fig. 1. In the parametric (α,β)-plane of Fig. 2, it is evident the effect of the noise intensity D on the transition boundary between the appearance of single and multi-limit cycles orbits: the bifurcation that leads to birhythmicity is postponed under the influence of noise.[35] As a consequence, the region of existence of three limit cycles, a condition for birhythmicity, decreases with the increase of the noise intensity and disappears altogether for high noise intensity.

An important feature of birhythmicity in the present model is highlighted in Fig. 3. We first define $\mathcal{P}_{1,3}$ of the probability to find the system in the basin of attraction of each stable orbits 1 and 3

$$\mathcal{P}_1 = \int_0^{A_2} P(A)dA,$$
$$\mathcal{P}_3 = \int_{A_2}^\infty P(A)dA. \tag{18}$$

These quantities measure the relative stabilities pertaining to the attractors 1 and 3 and are related to the resident time by the relation $\mathcal{P}_{1,3} = T_{1.3}/(T_1 + T_3)$, so that $\mathcal{P}_1/\mathcal{P}_3 = T_1/T_3$. In Fig. 3, we show in the parameter plane α–β the locus where the system stays with equal probability on both
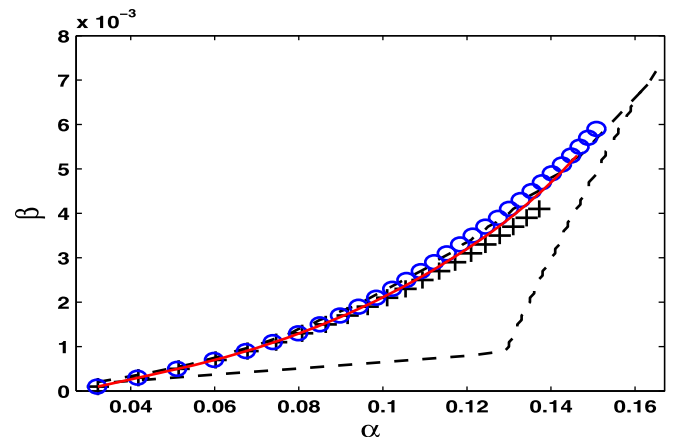

FIG. 3. Behavior of the residence times in the parameter space. The solid line denotes the locus $T_1 = T_3$, while circles and crosses denote the situation where $T_1 = 10T_3$ and $T_1 = T_3/10$, respectively. The dashed line is the border of existence of birhythmicity. The noise level is D = 0.1.

attractors (solid line) $T_1 = T_3$. We also show two further curves: the limit where the first attractor is much more stable than the other ($T_1 \geq 10T_3$, circles) and the passage to the reverse situation ($T_3 \geq 10T_1$, crosses). From the figure, it is evident that the outer attractor is dominantly visited in most of the parameter plane. Moreover, the transition from the two opposite cases (i.e., a change of two order of magnitudes of the relative resident times) occurs with a very narrow change of the control parameters α and β. The drastic change is further investigated in Fig. 4, where we show a blow-up of the crossover region around $T_1 = T_3$ for different values of the parameter β. The α value is increased up to the maximum value when birhythmicity disappears. The general behavior observed for all β values closely reminds phase transitions: the probability to find the system in one condition (around the attractor A_1) or the other (around the attractor A_3) drastically changes in a very small interval of the α parameter. The same behavior, this time with a constant value of α and varying β is shown in Fig. 5. The effect of the noise intensity is much less pronounced, see Fig. 6. It is apparent that the

effective temperature is capable to cause a crossover between the residence times only in a very narrow region of the phase space, inasmuch as the noise causes a contraction of the region of existence of birhythmicity. However, it is evident that the transition is much slower, and a crossover only occurs in the limited parameter space around $\alpha = 0.05$, $\beta = 0.0005$.

The stability properties of the two attractors have also been investigated in the limit of small noise values,[6] where it has been found the same asymmetrical behavior of the probability distribution, with a sudden change for small variations of α and β. In fact one can notice that the effective *Langevin* equation (10) amounts to the Brownian motion of a particle in a double well, whose potential reads:[20]

$$\dot{A} = -\frac{\partial F_A(A)}{\partial A} - \sqrt{\frac{D}{2w_0^2}} \langle \Gamma(t) \rangle,$$

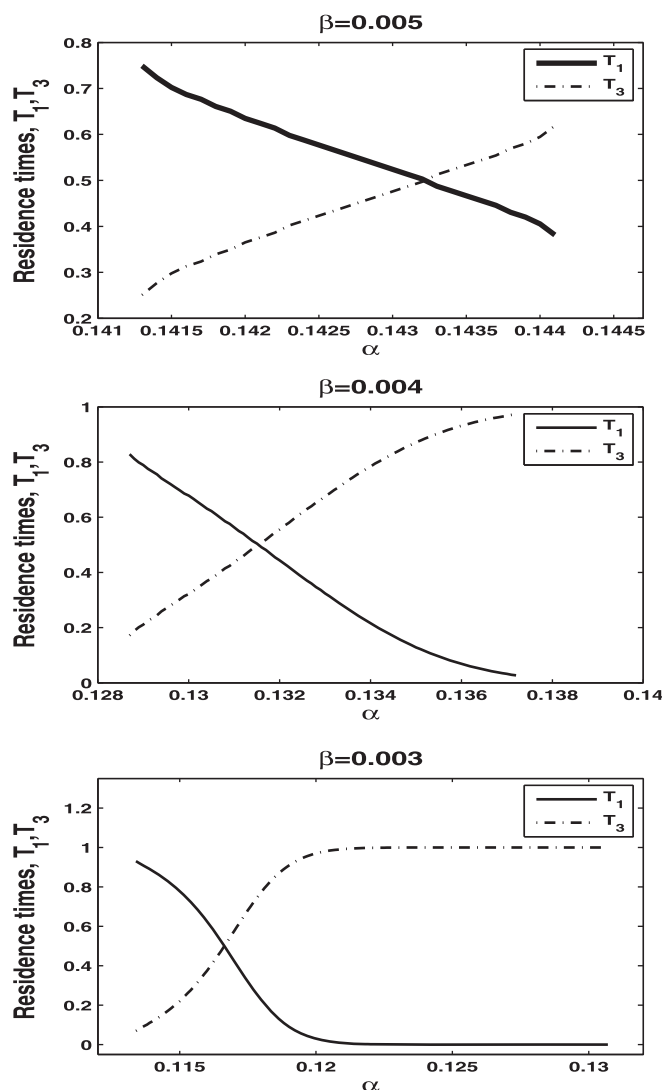$$F_A(A) = -\frac{\mu A^2}{4} \left[\left(1 - \frac{1}{8}A^2 + \frac{1}{24}\alpha A^4 - \frac{5}{256}\beta A^6 \right) \right]. \tag{19}$$
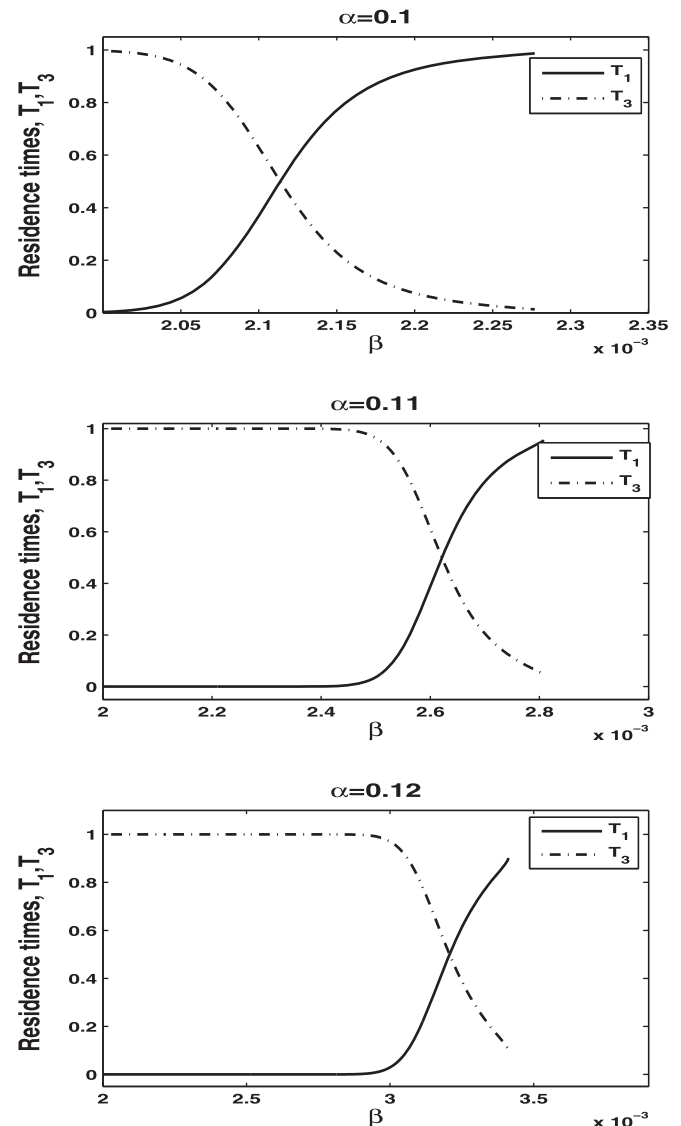

FIG. 4. Residence times as a function of the parameter α for different values of the parameter β. The noise level is D = 0.1, the nonlinearity $\mu = 0.1$.

FIG. 5. Residence times as a function of the parameter β for different values of the parameter α. The noise level is D = 0.1, the nonlinearity $\mu = 0.1$.
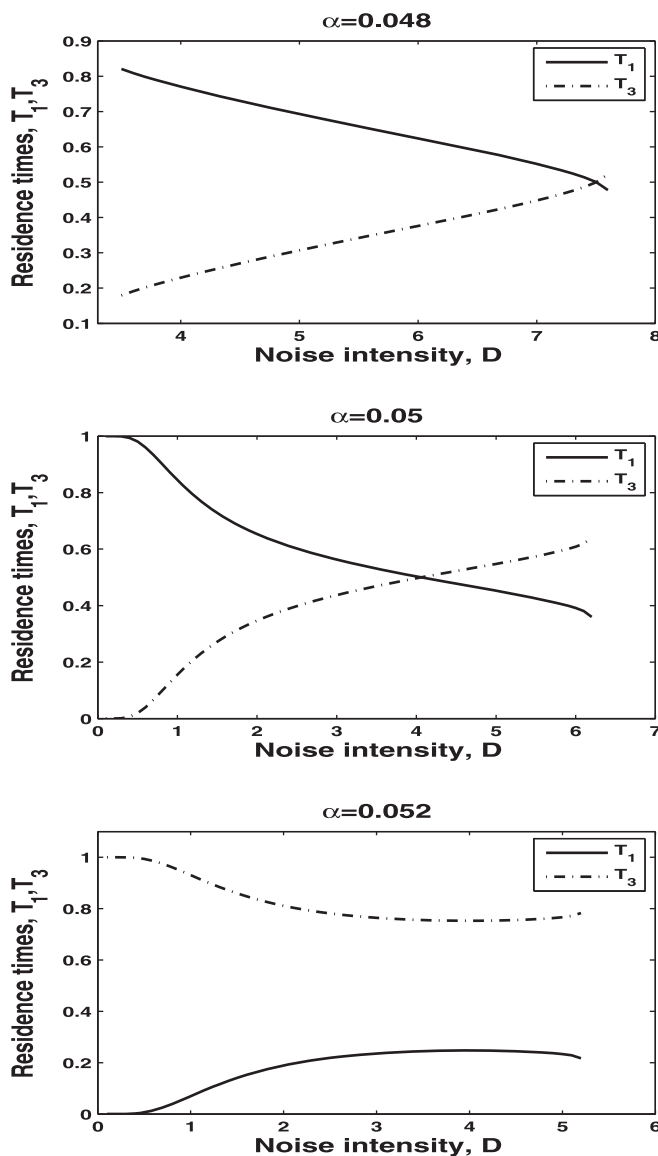
FIG. 6. Residence times as a function of the noise intensity D for different values of the parameter α. The second dissipation parameter reads $\beta = 0.0005$, the nonlinearity $\mu = 0.1$.

It is therefore evident that the transition from the inner orbit A_1 to the outer orbit A_3 through the unstable orbit A_2, as well as the inverse process, can be interpreted as the diffusion over an effective potential barrier, and therefore the escape times are given by the *Kramer's* inverse rate,[36] for instance, used in Josephson physics to detect the classical quantum/classical crossover[37] or for signal detection.[38]

$$\tau_{1\to3} \propto \exp\left[\frac{4\Omega^2}{D}\left(F_A(A_2) - F_A(A_1)\right)\right] = \exp\left[\frac{\Delta U_1}{D}\right],$$

$$\tau_{3\to1} \propto \exp\left[\frac{4\Omega^2}{D}\left(F_A(A_2) - F_A(A_3)\right)\right] = \exp\left[\frac{\Delta U_3}{D}\right]. \quad (20)$$

Thus, the average time to pass from one attractor to the other is analogous to the passage over a barrier. The pseudopotential barrier numerically derived in Ref. 6 is therefore, in the phase-amplitude approximation, an effective potential for

the amplitude variables.[20] Since the effective potential is analytical, we can confirm several features of the pseudo-potential, for instance, that the potential barriers are proportional to the nonlinear parameter μ.[6] It is also interesting to investigate the behavior of the potential barriers of Eq. (20) as a function of the parameters α and β, the analogous of the analysis of Eq. (16) in Figs. 3–6. Inspection of the effective potential (20) confirms that it is very asymmetrical, since one energy barrier is generally much higher than the other. Combining this observation with the exponential behavior of the escape rates (20) one deduces that the system does not equally stays on both attractors, but rather it clearly exhibits a preference for one attractor with respect to the other (the relative occupancies read $T_1/T_3 \simeq \exp[(\Delta U_3 - \Delta U_1)/D]$[36]). One concludes that the birhythmic system behaves as a bistable tunnel diode[36]: keeping fixed a control parameter (say β) and changing the other (α in this case) the weight of the probability distribution is concentrated in the proximity of one or the other of the two stable deterministic solutions of Eq. (1), thus obtaining again a first order phase transition. This result supports the notion that the analogy with phase transitions is generic for bistable oscillators.[39]

IV. NUMERICAL SIMULATIONS AND RESULTS

To check the validity of the approximations behind the analytic treatment that has led to the solution (16), we have performed numerical simulations of the Langevin dynamics (2). There are several methods and algorithms for solving second-order stochastic differential equations[40] as the implicit midpoint rule with *Heun* and *Leapfrog* methods or faster numerical algorithms such as the stochastic version of the *Runge-Kutta* methods and a quasisymplectic algorithm.[41] To prove that the simple procedure given by the *Euler* algorithm is reliable, we have employed it in a few selected points with two different methods. The starting point is the *Box-Mueller* algorithm[42] to generate a Gaussian white noise distributed random variable $\Gamma_{\Delta t}$ from two random numbers a and b which are uniformly distributed on the unit interval [0, 1]. The random number approximates the effect of the noise of intensity D over the interval Δt in the *Euler* algorithm for the integration of Eq. (8). We have then halved the step size Δt until the results became independent of the step size; the step size used for all numerical integration is $\Delta t = 0.001$. To verify the numerical results obtained with the Euler method, we have used a quasi-symplectic algorithm of *Mannella*[41] to numerically compute the probability distribution. The logic behind the choice to compare the *Euler* algorithm with a quasi-symplectic algorithm is that the nonlinear dissipation of the model (2) oscillates and vanishes twice in each cycle. We have therefore checked the results with an algorithm that has proved to perform independently of the dissipation value.[40]

In Fig. 7, we plot the behavior of the probability distribution P as a function of the amplitude A for several values of the noise intensity D, when the frequencies of both attractors are similar, i.e., $\Omega_1 \simeq \Omega_3 \simeq 1$. It clearly shows that the system is more likely found at two distinct distances from the origin, the essential feature of birhythmicity. In general, for the set of parameters $\alpha = 0.083$, $\beta = 0.0014$, the probability
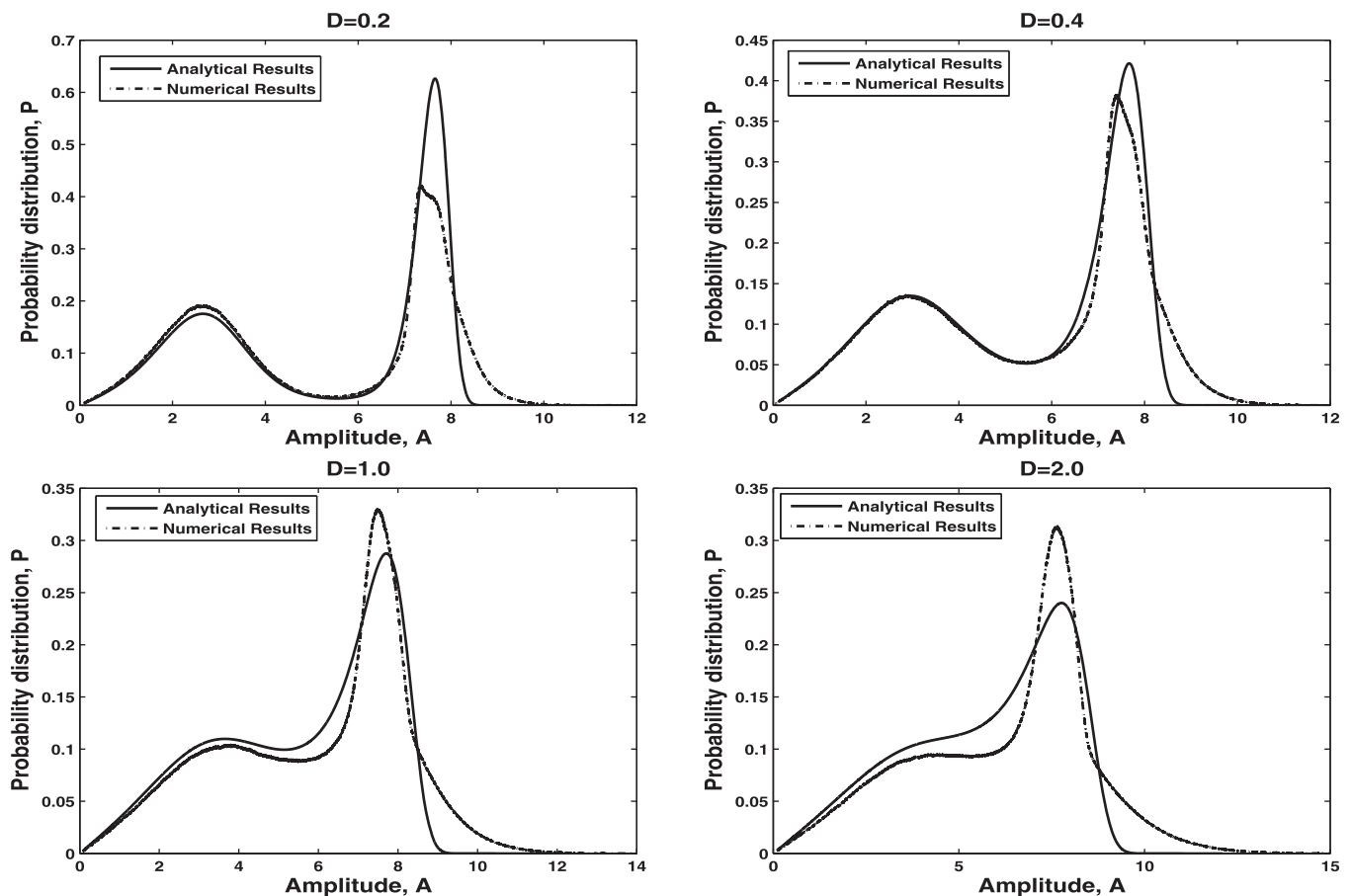
FIG. 7. Asymmetric probability distributions for different values of the noise intensity D versus the amplitude A when the frequencies of both attractors are identical, i.e., $\Omega_1 \simeq \Omega_3 \simeq 1$. Parameters of the system are $\mu = 0.1$ and $\alpha = 0.083$, $\beta = 0.0014$.

distribution P is asymmetric. As observed in Sec. III, the probability distribution changes with a small variation of the parameters α and β.[6,20] It is important to note that the agreement between numerical and analytical results is fairly good for low A values around the inner orbit, when the frequency of one attractor is very similar to 1, while for larger amplitude A the agreement becomes progressively poorer. However, it seems that the phase-amplitude approximation is capable to capture the main feature of the phenomenon: an increase or a decrease of the amplitude when the fluctuation parameter D is varied. At high fluctuations ($D > 1$) the system becomes monorhythmical, see also Fig. 2, thus confirming the noise induced transition from bimodal to unimodal, sometimes referred to as phenomenological bifurcations.[18]

As mentioned in Sec. III, the phase-amplitude approximation is not appropriate when the two frequencies of the attractors are different, i.e., $\Omega_1 \neq \Omega_3$. In fact numerical simulations in this case show a poor agreement, see Fig. 8 where $\Omega_1 \simeq 1$ and $\Omega_3 \simeq 0.8$. This shows the limitations of this analysis of phase-amplitude approximation.

Let us return to Eq. (17) that shows how the orbits radii depend on the noise intensity D. The analytical and numerical behaviors of the limit cycle attractors are reported in Figs. 9 and 10 that show amplitudes $A_{1,3}$ and the associated bandwidths $\Delta A_{1,3}$ (the width when the height of the probability peaks is reduced of a factor 2) as a function of the noise intensity D for two sets of parameters α and β. We find that the
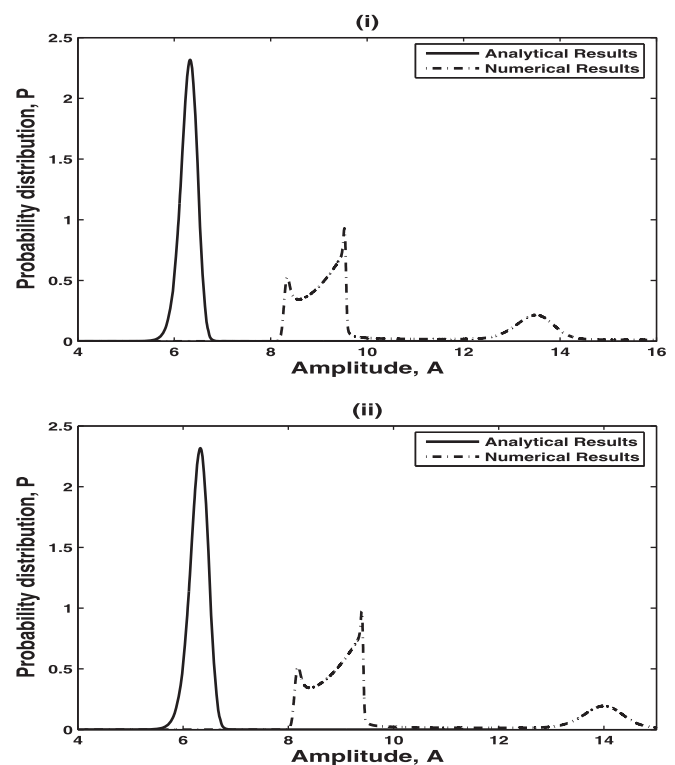

FIG. 8. Probability distribution versus the amplitude A when the frequencies of the attractors are not identical i.e $\Omega_1 \neq \Omega_3$. Parameters of the system are D = 0.1, $\mu = 0.1$, (i): $\alpha = 0.09$, $\beta = 0.0012$, $\Omega_1 \simeq 1, \Omega_3 \simeq 0.85$ and (ii): $\alpha = 0.1, \beta = 0.014, \Omega_1 \simeq 1, \Omega_3 \simeq 0.8$.
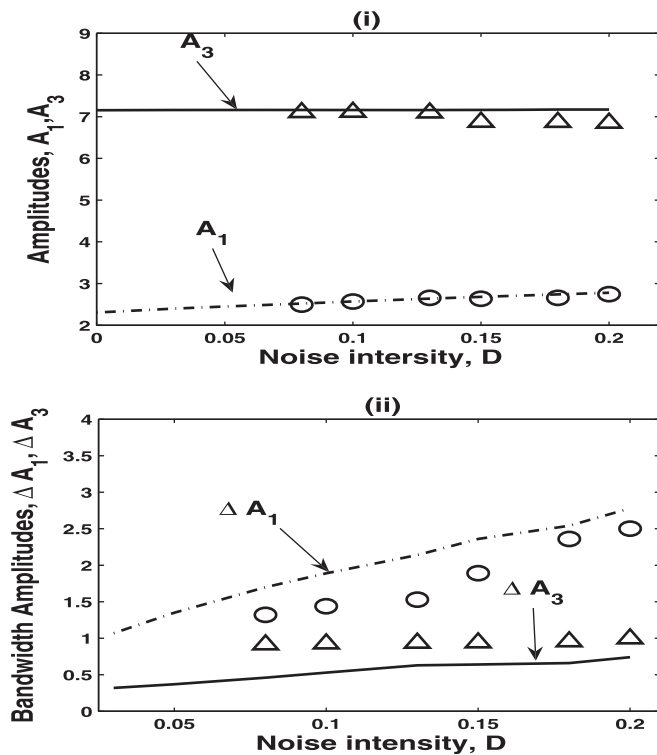
FIG. 9. Variation of the amplitudes A_i and the bandwidths ΔA_i versus the noise intensity D. Lines and symbols denote analytical and numerical results, respectively. The circles and dot-dashed lines refer to the inner attractor A_1, solid lines and triangles to the outer attractor A_3. The parameters used are $\mu = 0.1$, $\alpha = 0.1$, $\beta = 0.002$.
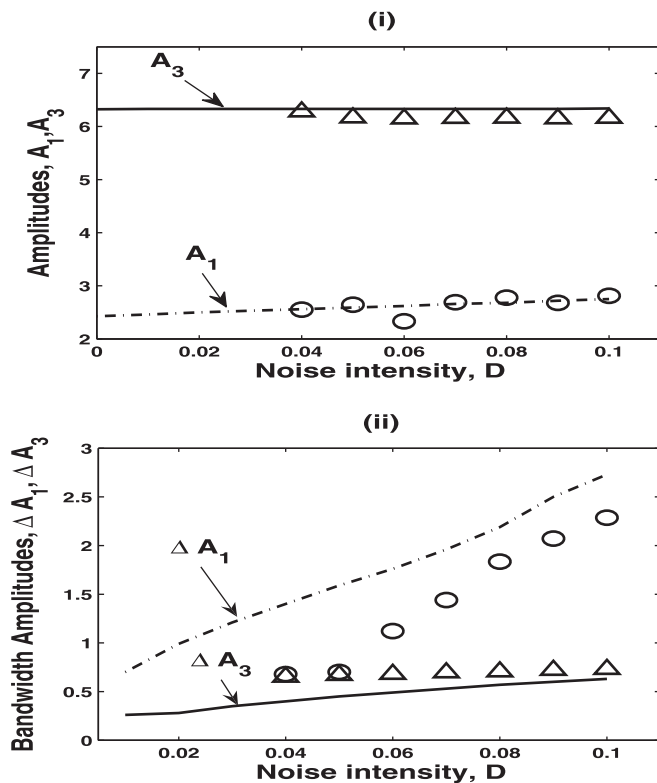

FIG. 10. Variation of the amplitudes A_i and the bandwidths ΔA_i versus the noise intensity D. Lines and symbols denote analytical and numerical results, respectively. The circles and dot-dashed lines refer to the inner attractor A_1, solid lines and triangles to the outer attractor A_3. The parameters used are $\mu = 0.1$, $\alpha = 0.12$, $\beta = 0.003$.
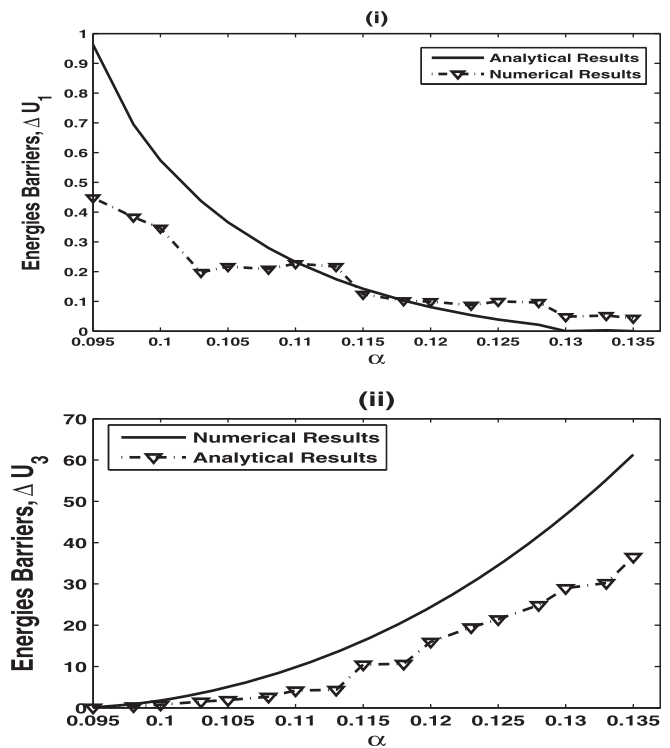

FIG. 11. Behavior of energy barriers versus α. Solid lines denote the analytical results, while dashed lines with triangles denote numerical simulations. Parameters of the system are $\mu = 0.1$ and $\beta = 0.002$.

amplitudes A_1 and A_3 change very slightly when the noise intensity increases. Also the bandwidth slightly increases with the noise intensity D. Through Eqs. (16) and (17) one can derive the behavior of the effective potential barriers.[6,20] We
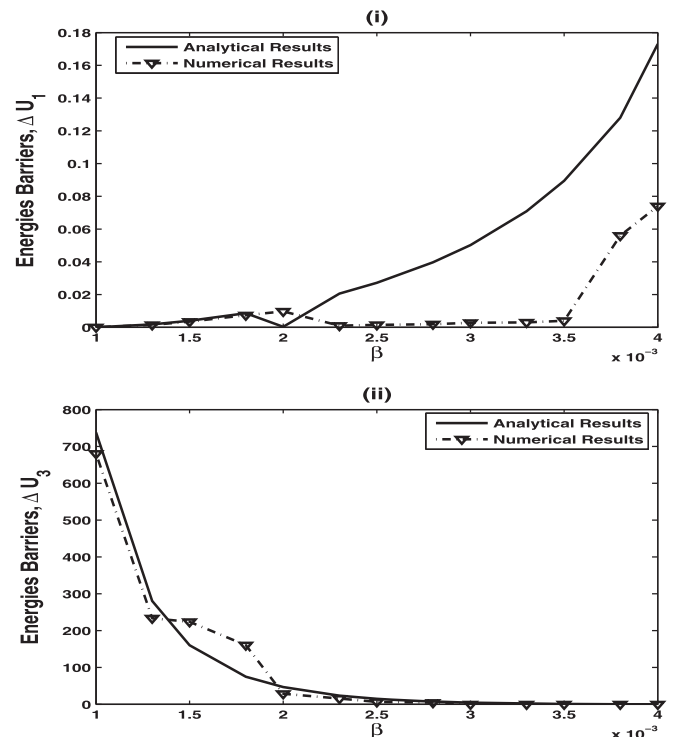

FIG. 12. Behavior of energy barriers versus β. Solid lines denote the analytical results, while dashed lines with triangles denote numerical simulations. Parameters of the system are $\mu = 0.1$ and $\alpha = 0.13$.

have numerically compared the analytic predictions with simulations in Fig. 11, where we plot ΔU_1 and ΔU_3 as a function of the parameter α with $\beta = 0.002$. It should be noted that according to the numerical results of Ref. 6, varying α from $\alpha = 0.095$ to $\alpha = 0.135$ the system passes from the region where $\Omega_1 \simeq \Omega_3$ to the region with $\Omega_1 \neq \Omega_3$. It is clear that in general the two energy barriers are very different. For low α values ΔU_3 is well approximated by the analytic approach, while for larger α the agreement becomes progressively poorer. Nevertheless, it seems that the phase-amplitude approximation is capable to capture the main feature of the phenomenon: an increase or a decrease of the activation energies when the dissipation parameters are varied. An analogous behavior is observed in Fig. 12, where we plot the behaviors of ΔU_1 and ΔU_3 as a function of the parameter β.

V. CONCLUSIONS

We have approached a theoretical description of the temporal evolution of the modified *van der Pol* oscillator with an additive Gaussian white noise in the region where birhythmicity (in the absence of noise) occurs. To get an analytical insight on this system we have used an explicit solution based on the phase-amplitude approximation of the *Fokker-Planck* equation to analytically derive the probability distributions. The activation energies associated to the switches between different attractors have been derived analytically and numerically. We have found that the agreement is fairly good. The characteristics of the birhythmic properties in a modified *van der Pol* oscillator are strongly influenced by both the nonlinear coefficients α, β and the noise intensity D. The boundary of the existence of multi-limit-cycle solutions, in the parametric (α, β)-plane, decreases with the increase of the noise intensity D. Finally, the analytic estimate of the stability of the two attractors varies with the control parameters (the dissipation α and β) in a way that resembles phase transitions: for most parameters value the system is located around only one attractor, the other being visited with a vanishingly small probability. Only at special values of the control parameters the residence times are comparable, in agreement with experimental observations of birhythmicity in Biological systems: the passage from an attractor to another only occurs by varying the external parameters and not under the influence of noise.[43,44]

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