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Synchronization Dynamics of Modified Relay-coupled Chaotic Systems

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Abstract

In this manuscript, we study the dynamics of a modified relay-coupled chaotic systems. The modification consists on the fact that the relay unit is modeled to lead the entire network to a desired dynamics. Then we achieve finite-time synchronization indirectly through a linear combination of the three systems. Further, we consider the existence of a switch on time of the coupling from the relay unit to the outer systems. It appears some interesting behaviors such as bifurcations, alternation of crisis and phases transitions when varied the switch on time. An open result is also found. In our scheme and for the selected changeable initial conditions, it seems that the appearance or disappearance of coexistence of attractors is linked to the type of synchronization we are dealing with. Mathematical demonstrations are given to sustain our theory while numerical simulations show its effectiveness.

1 Introduction

In Ref [1], the authors said that: *An elegant way to enhance synchronization is the use of a relay unit between the systems to be synchronized.* In that paper, the authors gave a definition of relay synchronization as *a complete synchronization (CS) of two dynamical systems by indirect coupling through a relay unit, whose dynamics does not necessary join the synchronous state.* The interest for studying relay-coupled systems comes from multiple applications in science and engineering. In 2004 E. Camacho et al. [2] were motivated by the presence of circadian rhythms in the chemistry of the eyes. Although there was not direct connection between the two eyes in their model, they could mutually influence each other by affecting the concentration of melatonin in the bloodstream which represented the relay unit(that they called the bath). In ref. [1] the authors claimed that *This type of network module can be expected to exist, for instance, within the complex functional architecture of the brain.*

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In addition, studying the collective behavior of the chaotic oscillators with environmental coupling, C. Quintero-Quiroz and M. G. Cosenza in ref. [3] stated that *Examples of such systems include chemical and genetic oscillators where coupling is through exchange of chemicals with the surrounding medium.*

In almost all papers on this topic, the authors are concentrated only on the investigation of the behaviors of the relay-coupled systems simply by varying some parameters of the systems [1–5]. In the present manuscript, we plan to generate the form or the mathematical model of the relay unit that will always lead us to any desired behavior, for instance finite-time synchronization. For more synchronization techniques, the drive and response systems or networks involved respond for time tending to infinity. These methods are generally classified as asymptotic stability based synchronization [6–11]. Trying to attain fast convergence speed, powerful methods, including finite-time synchronization, have been introduced and applied [12–16], where finite time synchronization refers to the identical evolution of the interacting systems at a finite established time [17].

We think that to consider asymptotic stability based synchronization is far from the reality for some systems, since they present a perfect synchronization within a finite time (example of eyes). In addition, the finite-time stability is not an illusion since it is well applied in the domain of biological systems for example (See refs. [18, 19]). In ref. [18] H. B. Oza et al. proposed a modified version of the Michaelis-Menten function which led them to the possibility of modeling the naturally occurring finite-time behavior of the healthy immune system. Basing ourselves on such an idea, we propose to develop the expression of the relay units which can lead us to finite-time synchronization. To reach our goal, we show that for systems coupled through a relay, one can develop a relationship between the three systems such that relay synchronization becomes indirectly a finite-time relay synchronization.

This finite-time relay synchronization principle is that, it could exist a certain relationship between the three systems such that the newly obtained system is finite-time stable. Given three dynamical systems $\dot{x}(t) = f(x(t))$ and $\dot{z}(t) = f(z(t))$ the outer isolated systems and $\dot{y}(t) = f_0(y(t))$ the relay unit, we assume the existence of a fourth system $\dot{\zeta}(t) = g(\zeta(t))$ such that $\zeta(t) = x(t) - C_1 y(t) + z(t) - C_2 y(t)$, is Lipschitz continuous [20], where $C_i, i = 1,2$ are well chosen parameters which could be time dependent depending on the goals. $f_0(\cdot)$ is a new function derived from $f(\cdot)$. How is this derived will be shown in the appendix. This means that the expressions of $\dot{x}(t)$ and $\dot{z}(t)$ without any control inputs in one hand and the expression of $\dot{y}(t)$ in other hand are really different. According to what we said before, we will move from finite-time stability of the ζ' system to show the indirect finite-time synchronization of the entire network. This is clear since the ζ' system is a linear combination of the others.

In this paper, we focus on the chaotic Rössler-like oscillator [4, 5]. Some advantages of this relay synchronization scheme are that its stability could be easily shown and that the coupled systems and the relay unit behave exactly as we need. Thus, this modified version of relay-coupling based synchronization is useful due to its malleability. In addition, this study shows that the type of synchronization is linked to the sign of the bound of the time derivative of the Lyapunov function.

The manuscript is organized as follows: in section 2, the theoretical analysis of the proposed scheme is presented. We show that the derived system could be finite-time stable and we derive the expression of the time of synchronization. Sections 3 and 4 are devoted to the dynamics of the entire coupled systems in the finite-time synchronization and practical synchronization domains, respectively. Phenomena such as interior and exterior crisis, intermittencies, phase synchrony, are investigated. We conclude in the last section.

2 Theoretical analysis and stability

To present our strategy, we focus on the synchronization of three systems of Rössler oscillators [4] coupled in an open-chain configuration shown by Fig. 1. The outer systems are $X(\tau)$ and $Z(\tau)$ while the relay unit is $Y(\tau)$. The switches T_{C1} and T_{C2} are used to control the activation times of the couplings

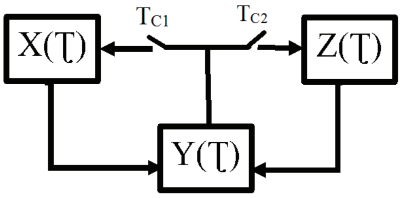

Fig. 1 Schematic representation of the relay connection. T_{C1} and T_{C2} are used to control the activation times of the couplings in both directions $Y(\tau) \to X(\tau)$ and $Y(\tau) \to Z(\tau)$ respectively.

in both directions $Y(\tau) \to X(\tau)$ and $Y(\tau) \to Z(\tau)$ respectively. The connection between the outer systems is made through the relay which is not necessarily identical to them.

We shall follow Sharma et al. [4] who studied chaotic oscillators coupled in relay in absence of time delay, with a series of synchronization phenomena. In the remaining of this paper, we shall study them from the view point of finite-time synchronization. The system consists of Rössler oscillators where the outer systems are described by:

$$\dot{\rho}_1 = -\omega\rho_2 - \varepsilon\left(\rho_1^2 + \rho_2^2\right)\rho_2 - \rho_3, \tag{1a}$$
$$\dot{\rho}_2 = \omega\rho_1 + a_1\rho_2 + \varepsilon\left(\rho_1^2 + \rho_2^2\right)\rho_1 - k_1(\tau)\left(\rho_2 - C_{1,2}y_2\right), \tag{1b}$$
$$\dot{\rho}_3 = a_2 + (\rho_1 - c)\rho_3 - k_2\left(\rho_1 - C_{1,2}y_1\right) - k_3\left(\rho_3 - C_{1,2}y_3\right), \tag{1c}$$

where $\rho(\tau)$ represents $X(\tau)$ or $Z(\tau)$, a_1, a_2, ω and c are system parameters and C_i, $i = 1,2$ are control parameters such that C_1 or C_2 is used when $\rho(\tau)$ is $X(\tau)$ or $Z(\tau)$ respectively while the coupling parameters $k_i, i = 2,3$ are used to maintain the stability of the coupled systems. $k_1(\tau)$ is the coupling function which leads to finite-time synchrony. This coupling function is switched on on time T_{C_1} or T_{C_2}, depending of the outer system being connected to the relay, they will be considered as equal for the time being and called T_C. Hence T_C becomes the time of activation of the finite-time based coupling and expressed as follows

$$k_1(\tau) = \begin{cases} 0 & \text{if } \tau \le T_C, \\ \eta\left(\theta - \dfrac{p}{\left(\rho_2 - C_{1,2}y_2\right)^2 + \varepsilon}\right) & \text{otherwise,} \end{cases} \tag{2}$$

where θ, η and p are constants to be defined by the designer and ε is a constant used to avoid the division by zero. In what follows we consider $T_C = 0$.

Our aim is to design the relay unit such that the outer systems $X(\tau)$ and $Z(\tau)$ synchronize in a predetermined time. To do so, we construct a new system ζ through the following relationship

$$\zeta = X(\tau) - C_1Y(\tau) + Z(\tau) - C_2Y(\tau), \tag{3}$$

such that the obtained ζ's system is finite-time stable. Thus, if the ζ's system is finite-time stable, the set of systems $X(\tau)$, $Y(\tau)$ and $Z(\tau)$ are also finite-time stable and then $X(\tau)$ and $Z(\tau)$ are finite-time synchronized.

Basing ourselves on Eq.3, the relay unit system is written as follows (refer to the calculation in appendix):

$$\dot{y}_1 = -\omega y_2 - y_3 - \frac{\varepsilon}{2}\left(x_1^2 + z_1^2 + x_2^2 + z_2^2\right)y_2 - \frac{\varepsilon}{2\left(C_1 + C_2\right)}\left(z_1^2 + z_2^2 - x_1^2 - x_2^2\right)\left(x_2 - z_2\right), \tag{4a}$$
$$\dot{y}_2 = \omega y_1 + a_1 y_2 + \frac{\varepsilon}{2}\left(x_1^2 + z_1^2 + x_2^2 + z_2^2\right)y_1 + \frac{\varepsilon}{2\left(C_1 + C_2\right)}\left(z_1^2 + z_2^2 - x_1^2 - x_2^2\right)\left(x_1 - z_1\right), \tag{4b}$$
$$\dot{y}_3 = \frac{2a_2}{C_1 + C_2} - cy_3 + \frac{1}{2}\left(x_1 + z_1\right)y_3 + \frac{1}{2\left(C_1 + C_2\right)}\left(x_1 - z_1\right)\left(x_3 - z_3\right). \tag{4c}$$

If we suppose that the systems $X(\tau)$ and $Z(\tau)$ are synchronized then $(\rho_2 - C_{1,2}y_2)^2 = \frac{\zeta_2}{4}$. Thus, the new system in $\zeta(\tau)$ is :

$$\dot{\zeta}_1 = -[\omega + \frac{\varepsilon}{2}(x_1^2 + z_1^2 + x_2^2 + z_2^2)]\zeta_2 - \zeta_3, \tag{5a}$$

$$\dot{\zeta}_2 = [\omega + \frac{\varepsilon}{2}(x_1^2 + z_1^2 + x_2^2 + z_2^2)]\zeta_1 + (a_1 - k_1(\tau))\zeta_2, \tag{5b}$$

$$\dot{\zeta}_3 = -k_2\zeta_1 + [\frac{1}{2}(x_1 + z_1) - c - k_3]\zeta_3. \tag{5c}$$

Let us show that the system Eq.(5) could be finite-time stable. We consider the following Lyapunov function:

$$V(\tau) = \frac{1}{2}(\zeta_1^2 + \zeta_2^2 + \frac{\zeta_3^2}{|k_2|}), \tag{6}$$

Differentiating the Lyapunov function $V(\tau)$ with respect to time yields

$$\begin{aligned} \dot{V}(\tau) \leq & -(\eta\theta - a_1)\zeta_2^2 + \frac{4\eta p}{\zeta_2^2 + \varepsilon}\zeta_2^2 \\ & - \frac{\zeta_3^2}{k_2}[k_3 + c - Max(|\frac{1}{2}(x_1 + z_1)|)], \\ & \quad \text{If} \quad k_3 \geq Max(|\frac{1}{2}(x_1 + z_1)|) \\ & \quad \text{and} \quad \eta\theta \geq a_1 \\ \leq & 4\eta p. \end{aligned} \tag{7}$$

It comes from here that the system Eq.(5) is finite-time stable if the control parameter p is negative. Thus the stability time can be derived as in [21] by:

$$\tau_S \leq \tau_0 + \frac{V(\tau_0)}{4\eta p}. \tag{8}$$

The expression $\dot{V} < 4\eta p$ with $p \neq 0$ suggests that there are two domains of a possible existence of synchronization. If p is negative, we deal with the finite-time synchronization domain. However, if p is positive the coupled oscillators could be practically synchronized. Then, what will be the dynamics of the relay-coupled systems in both domains? This is the topic of investigation in the next sections.

3 Finite-time synchronization domain $p < 0$

For all subsections, we use the following initial conditions and system's parameters: $(x_1, x_2, x_3) = (0.4, 0.9, 0.1)$, $(y_1, y_2, y_3) = (0.41, 0.89, 0.11)$ and $(z_1, z_2, z_3) = (0.34, 0.93, 0.12)$, $a_1 = 0.15$, $a_2 = 0.4$, $\omega = 0.41$ and $c = 8.5$. In this section, the switches T_{C1} and T_{C2} have no significative effects then we consider them to be zero.

3.1 Synchronization and anti-synchronization

First, we choose the values of parameters $\eta = 1$, $p = -0.001$, $k_2 = -1$ and $k_3 = 10$ and $\theta = 1.5$ and $C_i = 1, i = 1, 2$. The Fig. (2)(a) and (b) give the chaotic attractors of the $X-$ oscillator and the relay unit in the plane (x_2, x_1) and (y_2, y_1) respectively, while the graphs on Fig. (2)(c) and (d) show the synchronization between all the systems. However, if we consider a negative values of parameters $C_i, i = 1, 2$, we observe the synchronization between the outer systems and the anti-synchronization between outer system and relay unit (see Fig. (2(e)-(f)).

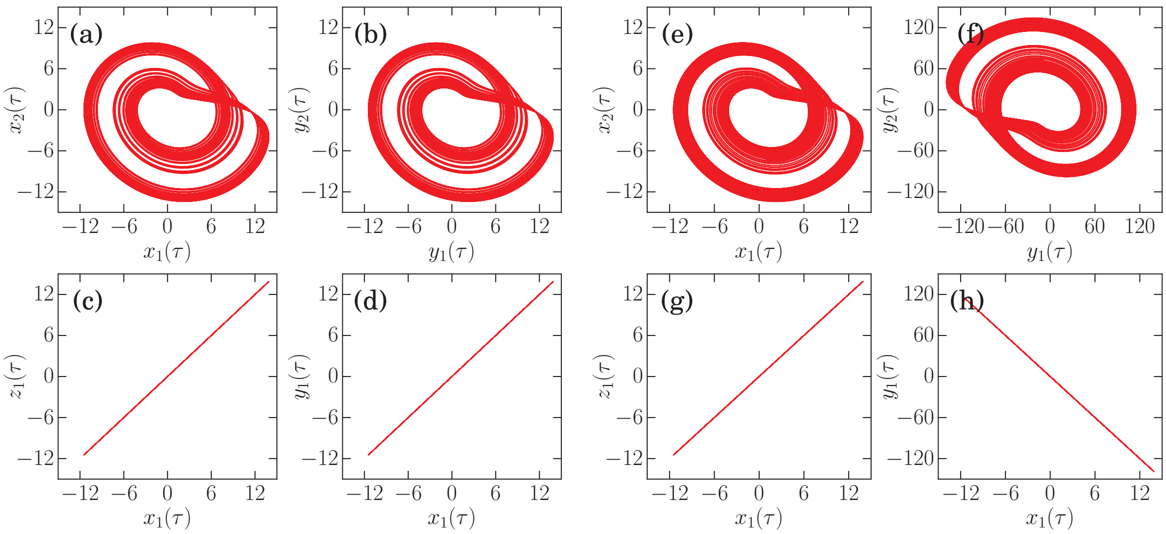

Fig. 2 Chaotic attractors in planes (a) (x_1,x_2) and (b) (y_1,y_2), (c) synchronization between the outer systems and (d) synchronization and reduction between outer system and relay unit with $C_1 = C_2 = 1$, $k_1 = 1$, $k_2 = -1$ and $k_3 = 10$. (e) (x_1,x_2) and (f) (y_1,y_2), (g) synchronization between the outer systems and (h) anti-synchronization with amplification between outer system and relay unit, with $C_1 = C_2 = -0.1$, $k_1 = 1$, $k_2 = -1$ and $k_3 = 10$.

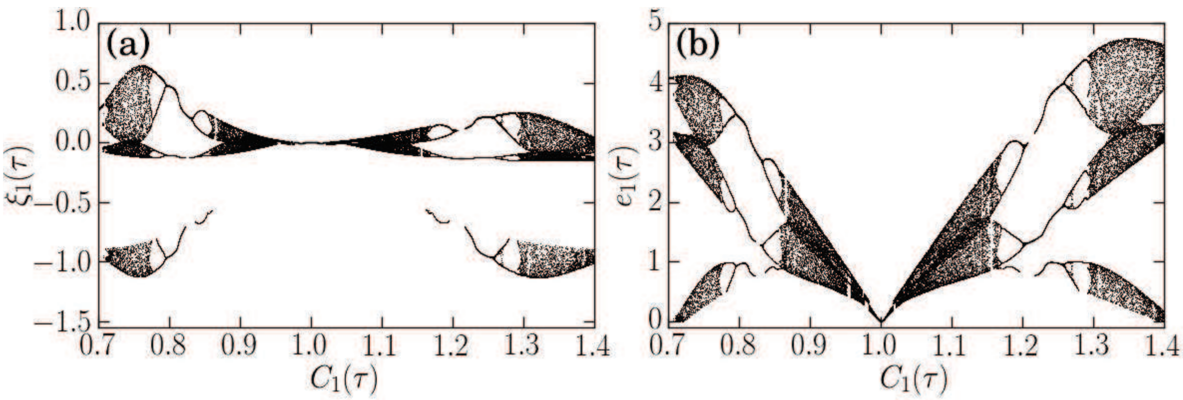

Fig. 3 Bifurcation diagrams for (a) ξ_1 and (b) $|z_1(\tau) - x_1(\tau)|$ as a function of the parameter C_1 with $C_2 = 1$.

Through simulations, we find that it appears a certain proportional relationship between the relay unit with the outer systems when they synchronize. This relation could be expressed as follows

$$y_1(\tau) = \Omega x_1(\tau), \tag{9}$$

where Ω is a constant. This recalls the works by Ioan Grosu et al. in [22] where they design a coupling for synchronization and amplification of chaos basing themselves on Lorenz systems. Let us now investigate the origin of this relationship in Eq.9. For this goal, let us consider the following bifurcation diagrams in Fig. 3.

The synchronization between the outer oscillators $X(\tau)$ and $Z(\tau)$ is obtained when $C_1 = C_2$ with $C_2 = 1$ since for that value of C_1, $\xi(\tau) = 0$ (see Fig. 3(a)). This conclusion is confirmed by the graph on Fig. 3(b) where the difference $e_1(\tau) = |z_1(\tau) - x_1(\tau)|$ is equal to zero at $C_1 = 1$.

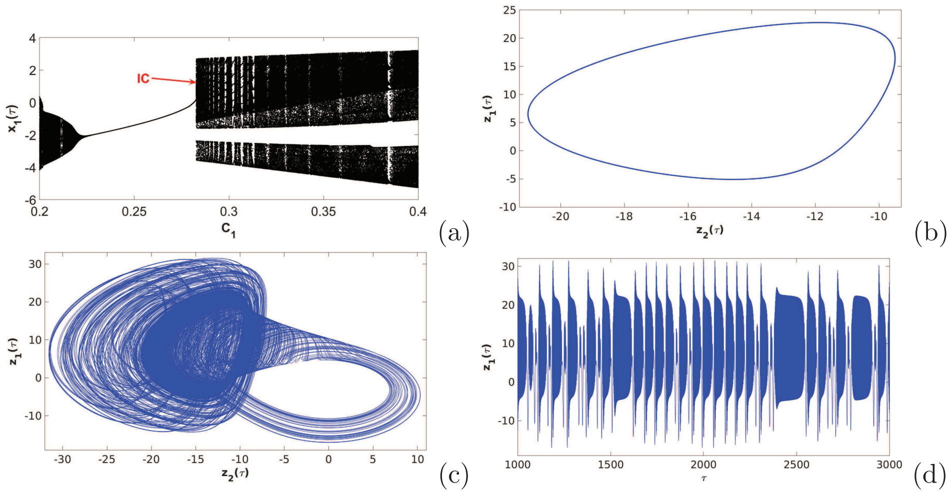

Fig. 4 Interior crisis and behaviors of the systems (a) Bifurcation diagrams for $x_1(\tau)$ as a function of the parameter C_1 with $C_2 = 1$. The crisis is determined at around $C_1 = 0.2829$ with $C_2 = 1$. (b) Limit cycle with one period of the system in the plane $(z_2(\tau), z_1(\tau))$ for $C_1 = 0.2829$ with $C_2 = 1$ (c) and (d) Chaotic attractor in the plane in the plane $(z_2(\tau), z_1(\tau))$ and Time history of $z_1(\tau)$ presenting intermittencies between two form of chaotic signals for $C_1 = 0.2830$ with $C_2 = 1$.

Let us consider the relation Eq.3, it comes out that

$$Y(\tau) = \frac{X(\tau) + Z(\tau) - \zeta(\tau)}{C_1 + C_2}. \tag{10}$$

According to Fig. 3, synchronization between $X(\tau)$ and $Z(\tau)$ means $\zeta(\tau) = 0$ and $C_1 = C_2$. Thus Eq.10 becomes

$$Y(\tau) = \frac{X(\tau))}{C_{1,2}}. \tag{11}$$

This implies that the constant of proportionality is expressed as follows

$$\Omega = {C_{1,2}}^{-1}. \tag{12}$$

We show that C_i with $i = 1, 2$ lead to finite-time synchronization if they are equal(with $p < 0$). However, if they are not equal, the coupled systems show among many behaviors crisis-induced-intermittency.

3.2 Interior crisis and crisis-induced-intermittency

Even if $p < 0$, changing the value of C_1 with $C_2 = 1$ leads the systems through some behaviors such as chaos, regularity and so on. Without getting into more details, we focus on the appearance of the crisis and crisis-induced-intermittency. Crisis is a particular phenomenon described by the sudden appearance or disappearance of a strange attractor as the parameters of the system are changing [23]. In particular, at the interior crisis or the second type of crisis the size of the chaotic attractor suddenly increases [23]. In addition crisis-induced intermittency is described by the irregular alternation of phases of different forms of chaotic dynamics [24, 25]. In our study, it appears that around the value $C_1 = 0.2829$ the systems present an interior crisis (see Fig. 4(a)) which is described by a transition from one period limit cycle at $C_1 = 0.2829$ (regularity Fig. 4(b)) to the crisis-induced-intermittency $C_1 = 0.2830$ Figs. 4(c) and (d). We shall see later on that they also appears in Fig. 5.

Contrary to the other works on relay-coupled systems such as in refs. [4, 5, 26], some interesting properties of this scheme are: the simple and well defined control of the amplitude of the coupled

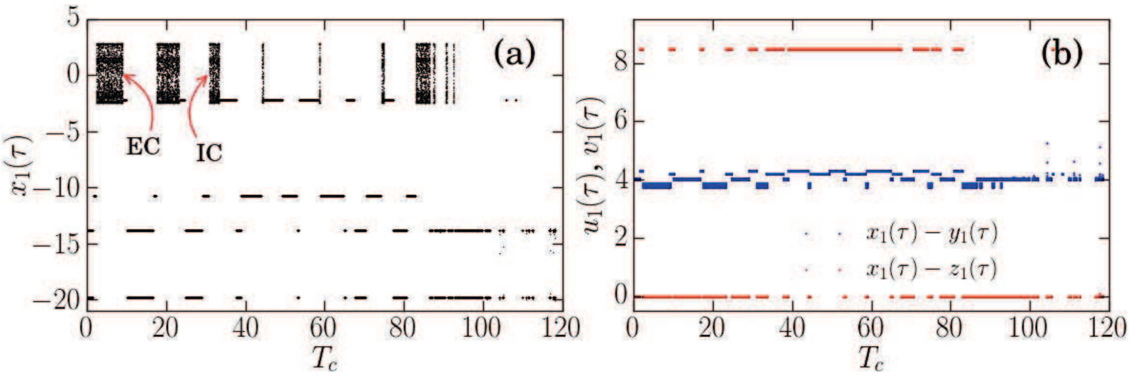

Fig. 5 Intermittent behaviors between chaos and regularity of the systems versus T_C. a) The system presents many external (EC) and interior (IC) crisis. b) Synchronization $(x(\tau)-z(\tau)=0)$ and out-of-synchronization $(x(\tau)-z(\tau)\neq 0)$ between outer systems $X(\tau)$ and $Z(\tau)$ (black dots) and out-of-synchronization between the outer system and the relay unit.

systems constituting the network (Eqs.(1) and (4)) by the husbandry of the parameters $C_i, i=1,2$ (See Eq.(11)) and the control of the synchronization time (See Eq.(8)).

4 Practical synchronization domain $p>0$

In general, when the derivative of the Lyapunov function is positive, we consider that the system could not stabilize. However, as shown by some references as [13, 27] some coupled oscillators can achieve practical synchronization for which the time derivative of the Lyapunov function is bounded. This means that the errors between the systems to synchronize (in the case of synchronization) are not going down to zero but remain within a small volume [27]. Let us investigate the dynamics of our relay coupled Rössler systems when $p>0$ meaning $V>0$ and $\dot{V}>0$. In this second domain, the coupled systems are sensitive to the influence of the switch T_C of the coupling $k_1(\tau)$. In what follows we fix $\theta=0.15$ (See Eq.(2)).

4.1 Effects of the switch on time T_C

4.1.1 Bifurcation diagrams

As mentioned, the switch T_C is used to model the time of activation of the coupling between the relay unit and one or both outer systems (See Fig1 and Eq.(2)). The influence of the switch on time T_C is described by the the following bifurcation diagrams for the initial conditions $(x_1,x_2,x_3)=(0.4,0.9,0.1)$, $(y_1,y_2,y_3)=(0.41,0.89,0.11)$ and $(z_1,z_2,z_3)=(0.34,0.93,0.12)$ and the parameters $\eta=1$ and $p=0.01$. These graphs show the intermittencies between chaos and regularity (or between interior (IC) and exterior (EC) crisis) due to T_C Fig. 5(a). The synchronization between the outer systems is shown when $(x(\tau)-z(\tau)=0)$(see Fig. 5(b)).

4.1.2 Coexistence of attractors

One of the surprising phenomena in nonlinear dynamics is the coexistence of attractors. This phenomenon has been found in many systems [25, 28, 29]. We find that in the domain of practical synchronization, the systems are sensitive to variation in initial conditions. To confirm that, we use the following graphs with the selected two sets of initial conditions INC_1 : $(x_1,x_2,x_3)=(0.4,0.9,0.1)$, $(y_1,y_2,y_3)=(0.41,0.89,0.11)$ and $(z_1,z_2,z_3)=(0.34,0.93,0.12)$ and INC_2 : $(x_1,x_2,x_3)=(0.4,0.8,0.1)$, $(y_1,y_2,y_3)=(0.26,0.89,0.11)$ and $(z_1,z_2,z_3)=(0.34,0.93,0.22)$ and the parameters $\eta=1$ and $p=1$.

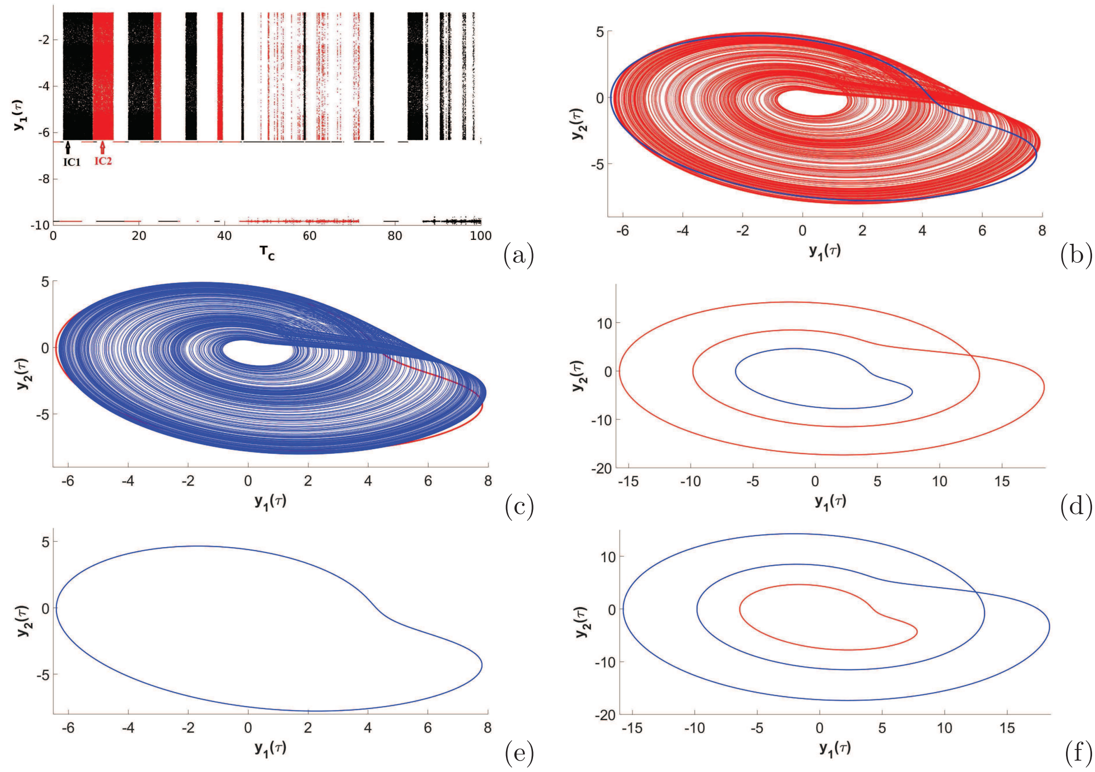

Fig. 6 Different behaviors between chaos and regularity of the systems as a function of T_C for the different sets of initial conditions. a) Two superimposed bifurcation diagrams completely different for the same range of T_C INC_1 in Black and INC_2 in red. We observe an exchange between the behaviors described by INC_1 chaotic implies INC_2 periodic or regular and vice versa. b) Coexistence of chaos (INC_1 in red) and period one limit cycle (INC_2 in blue) for $T_C = 23.33$. c) Coexistence of chaos (INC_2 in blue) and period one limit cycle (INC_1 in red) for $T_C = 10$. d) Coexistence of period two limit cycle (INC_1 in red) and period one limit cycle (INC_2 in blue) for $T_C = 27$. e) Identic behavior of the systems for the two set of initial conditions for $T_C = 34$. f) Coexistence of period two limit cycle (INC_2 in blue) and period one limit cycle (INC_1 in red) for $T_C = 50$.

The difference in behavior can be described by the bifurcation diagrams given in Fig. 6(a) where it appears an interchange between chaos for INC_1 and regularity for INC_2 and vice versa for a range of T_C. Some examples of coexistence are given by: chaos (INC_1 in red) and period one limit cycle (INC_2 in blue) for $T_C = 23.33$ (Fig. 6(b)), chaos (INC_2 in blue) and period one limit cycle (INC_1 in red) for $T_C = 10$(Fig. 6(c)), period two limit cycle (INC_1 in red) and period one limit cycle (INC_2 in blue) for $T_C = 27$(Fig. 6(d)) and period two limit cycle (INC_2 in blue) and period one limit cycle (INC_1 in red) for $T_C = 50$ (Fig. 6(f)). However, we found an identic behaviors of the oscillators for $T_C = 34$ in Fig. 6(e). Then according to these results, T_C could be used to control the coexistence phenomena from coexistence of different behaviors to identic behaviors.

The set of initial conditions, $x_2(0)$, $y_1(0)$ and $z_3(0)$, we choose to change with a view to observe coexistence was selected randomly. Also there are other values which give other limit cycles.

4.1.3 Dynamics of phases

In this subsection, we consider a mismatch between the two switches T_{C1} and T_{C2} such that the difference is $T_C + \tau_\eta$ and $T_{C2} = T_C$. This consideration is realistic and can find application in industry, telecommunication and so on.

To compute the oscillator phases, we use the same procedure as in ref. [4, 30, 31]. Let us consider

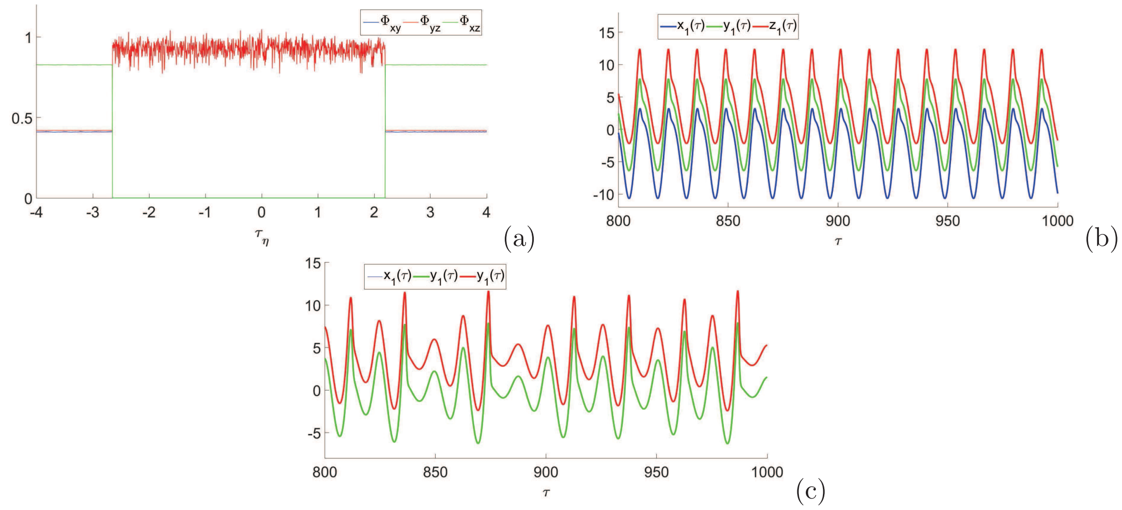

Fig. 7 Phases differences between the oscillators for $T_C = 5$, $\eta = 1$ and $p = 1$ a) The transitions from phase locked to in-phase synchrony and to phase locked of the outer systems. When the outer systems are in-phase synchrony they are out-of-phase synchrony with the relay oscillator. b) and c) Time histories of $x_1(\tau)$ (blue lines), $y_1(\tau)$ (green lines) and $z_1(\tau)$ (red lines) for $\tau_\eta = -5$ and $\tau_\eta = 1$ showing respectively the phase-locked between the systems and the synchrony of the outer systems associate to the out-of-phase synchrony with the relay.

an arbitrary signal $s(\tau)$ with its Hilbert transform $\tilde{s}(\tau)$ such that a complex function can be defined as

$$\psi(\tau) = s(\tau) + i\tilde{s}(\tau) = R(\tau)\exp^{i\phi(\tau)}, \tag{13}$$

where $R(\tau)$ is the amplitude and $\phi(\tau)$ the phase of the variable $s(\tau)$. If the instantaneous phase is $\phi_i(\tau)$, we can determine it through the following relation

$$\phi_i(\tau) = \tan^{-1}[\frac{\tilde{s}_i(\tau)}{s_i(\tau)}]. \tag{14}$$

The phase of each oscillator is constructed from the variable with subscript i and the average phase difference $\Phi_{ij}(\tau)$ between two oscillators is

$$\Phi_{ij}(\tau) = \langle|\phi_i(\tau) - \phi_j(\tau)|\rangle \quad for \quad i,j = x,y,z, \tag{15}$$

where $\langle\cdot\rangle$ denotes the time average.

Changing τ_η leads to phase synchronization between the outer systems when $\tau_\eta \in]-2.65, 2.195[$ as shown in Fig. (7). Outside this interval the systems seem to be phase locked [30]. In addition it is also interesting to see that the relay is out-of-phase synchrony with the outer systems when they synchronized.

Surprisingly, the switch on time mismatches τ_η can be used as chaos controller as shown by the bifurcation diagram and the Lyapunov exponent in Fig. 8 which show the evolution of the coupled systems from period one limit cycle to chaos and later period one limit cycle. The transition from regularity to chaos and vice versa are sudden which means that we are dealing with interior and boundary or exterior crisis. The exterior crisis is recognized by the sudden destruction of the attractor as the control parameter is varied [23]. In addition, the mutual influence of of τ_η and T_C on the phase synchrony is shown by Fig. (9). We find that that the synchronization is strongly linked to the values of parameters of the couple (τ_η, T_C).

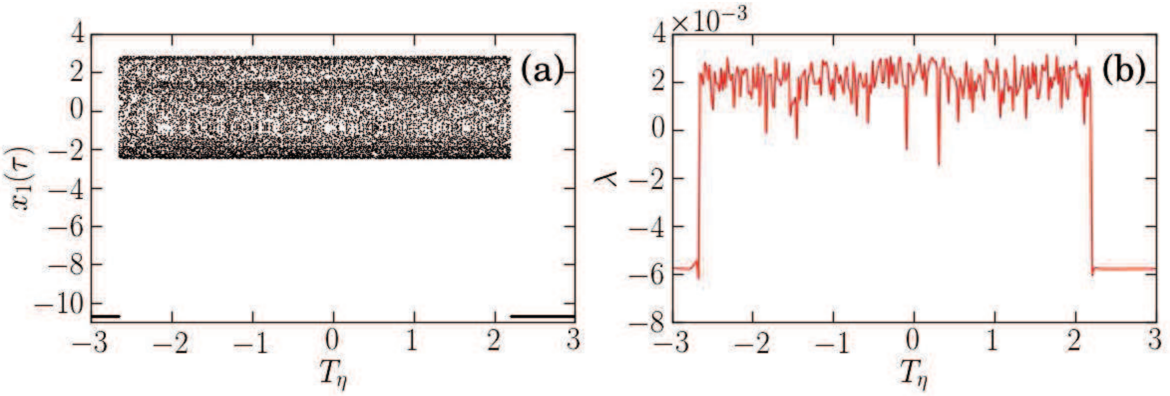

Fig. 8 (a) and (b) Bifurcation diagram and Lyapunov exponent of system $X(\tau)$ for $T_C = 5$, $\eta = 1$ and $p = 1$. The transitions from regularity to chaos and later regularity is shown and confirms the result on phase transition in Fig. (7).

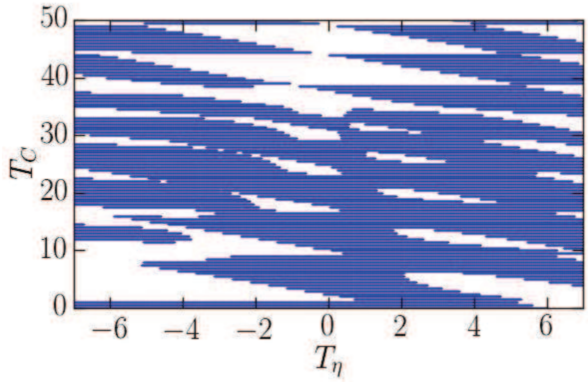

Fig. 9 Blue zone shows the couple (τ_η, T_C) for which the outer systems are in-phase synchrony.

4.2 Mutual effects of p and T_C

Let us focus on the mutual impact of the constants T_C (Meaning T_{C1} and T_{C2} are equal) and p since they are the real basis of our study in this manuscript. We express p as

$$p = 10^n, \quad \text{where} \quad n \in \mathbb{R}. \tag{16}$$

We fix $\eta = 1$ and varying T_C and n for the set of initial conditions IC_1. The graph in Fig. 10 gives the phase differences between the oscillators for the couple of parameters (n, T_C). Each color corresponds to a zone where at least two of the three systems are in phase: in red zone all the three systems are in-phase synchrony, only the oscillators $X(\tau)$ and $Y(\tau)$ synchronize in the blue area while only $Z(\tau)$ and $Y(\tau)$ synchronize in the yellow area and just the outer systems synchronize in phase in the green zone.

5 Conclusion

The target of this manuscript is to investigate the possibility to achieve synchronization in finite-time for a system of chaotic oscillators coupled through a relay unit. The investigations were carried out for

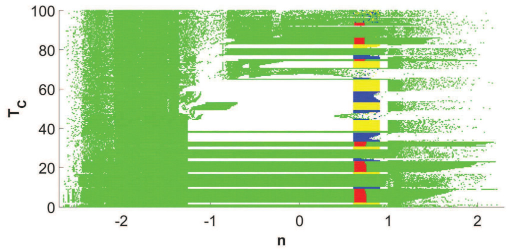

Fig. 10 Zones of phase synchronization: red zone all the three systems are in-phase synchrony, the oscillators $X(\tau)$ and $Y(\tau)$ synchronize in the blue area while $Z(\tau)$ and $Y(\tau)$ synchronize in the yellow area and The outer systems only synchronize in phase in the green zone.

chaotic Rössler-like oscillators [4]. The coupling was built in such that the newly designed ζ's system (Eq.(5)) becomes finite-time stable. The relationship between all systems recalls the one used in the case of projective or functional projective synchronization [32, 33]. Later we observed some interesting behaviors in the coupled systems such as: synchronization between outer systems while the relay unit is also synchronous with amplification of chaos, synchronization of outer systems while the relay unit anti-synchronizes with the others, oscillations death in ζ's system (Eq.(5)) which characterizes the complete synchronization between the outer systems, chaotic and non-chaotic behaviors of each system involving in the relay-coupled model. In addition, the synchronization, amplification or reduction status can be controlled through the parameters $C_i, i = 1, 2$.

Surprisingly for this chaotic Rössler-like oscillator we found that the coexistence phenomenon appears only if we are in the practical synchronization domain where the time derivative of the Lyapunov function is bounded by a positive constant. Also the switch T_C of the coupling or it's mismatch τ_η and the constant p of the coupling could be used to control the appearance of many behaviors such as intermittency, crisis, in-phase synchrony, chaos and so on.

Different from other works on relay-coupled systems, some interesting properties of this scheme are: the control of the amplitude of the relay unit by the husbandry of the parameters $C_i, i = 1, 2$, the control of the synchronization time. However, the mathematical development done to obtain the suitable form of the relay unit is tiresome in the general case. Also, the number of couplings used to achieve and maintain the finite-time stability depend on the expression of the considered systems as shown by Eqs.1 and .A1 or .A2

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APPENDIX

Appendix: Development of the relay unit: Simple case of a 4D- hyperchaotic system

In this paragraph, we are going to show the development of the relay unit expression. For simplicity, we base ourselves on the $4-D$ hyperchaotic system based on an extension of the diffusionless Lorenz system studied by Li and Sprott in [34] where the steps to calculate the relay system are similar to those of the Rössler system (Eq.1) although simpler. As said before, in this study the simplicity in development is linked to the expression and the number of the nonlinear terms contained in the used model. The $4-D$ hyperchaotic Lorenz system contains just two nonlinearities given as a products of two of its variables in its second and third equations. In detail the outer systems are given as follows:

$$\dot{x}_1 = x_2 - x_1, \tag{A1a}$$
$$\dot{x}_2 = -x_1x_3 + x_4 - k_1(\tau)(x_2 - C_1y_2), \tag{A1b}$$
$$\dot{x}_3 = x_1x_2 - a_1, \tag{A1c}$$
$$\dot{x}_4 = -a_2x_2, \tag{A1d}$$
$$\dot{z}_1 = z_2 - z_1, \tag{A2a}$$
$$\dot{z}_2 = -z_1z_3 + z_4 - k_1(\tau)(z_2 - C_2y_2), \tag{A2b}$$
$$\dot{z}_3 = z_1z_2 - a_1, \tag{A2c}$$
$$\dot{z}_4 = -a_2z_2, \tag{A2d}$$

where $k_1(\tau)$ is the coupling defined in Eq.(2). Now we search for a relation, Eq.(3), such that it fulfills the conditions of finite-time synchronization as defined in Section 2. It appears that $\dot{\zeta}_1 = \zeta_2 - \zeta_1$ and $\dot{\zeta}_4 = -a_2\zeta_2$,. The difficulty appears when dealing with nonlinearity. Here, the strategy is developed as follows:

We initiate the relay unit as follows

$$\dot{y}_1 = y_2 - y_1, \tag{A3a}$$
$$\dot{y}_2 = -\frac{1}{2}(x_1 + z_1)y_3 + y_4, \tag{A3b}$$
$$\dot{y}_3 = \frac{1}{2}(x_1 + z_1)y_2 - a_1, \tag{A3c}$$
$$\dot{y}_4 = -a_2y_2. \tag{A3d}$$

Let us develop the second and the third equations of the ζ's system. One has

$$\dot{\zeta}_2 = -k_1(\tau)\zeta_2 + \zeta_4 - x_1x_3 - z_1z_3 + \frac{C_1+C_2}{2}(x_1+z_1)y_3, \tag{A4a}$$
$$\dot{\zeta}_3 = x_1x_2 + z_1z_2 - 2a_1 + (C_1+C_2)a_1 - \frac{C_1+C_2}{2}(x_1+z_1)y_2. \tag{A4b}$$

If we add and subtract $\frac{1}{2}x_1z_3$ and $\frac{1}{2}x_3z_1$ in the first equation of system Eq.(A4) and $\frac{1}{2}x_1z_2$ and $\frac{1}{2}x_2z_1$ in its second equation, one has

$$\dot{\zeta}_2 = -k_1(\tau)\zeta_2 + \zeta_4 - \frac{1}{2}(x_1+z_1)\zeta_3 - \frac{1}{2}(x_1-z_1)(x_3-z_3), \tag{A5a}$$

$$\dot{\zeta}_3 = \frac{1}{2}(x_1+z_1)\zeta_2 + a_1(C_1+C_2-2) + \frac{1}{2}(x_1-z_1)(x_2-z_2). \tag{A5b}$$

At this stage, all term without any ζ have to be deleted in ζ's system. To do so, they have to be introduced into the corresponding equations of the dynamics of the relay unit after a division by the term $\frac{1}{C_1+C_2}$. Then we finally have the ζ's and the relay systems as given respectively by Eqs.(A6) and (A7).

$$\dot{\zeta}_1 = \zeta_2 - \zeta_1, \tag{A6a}$$

$$\dot{\zeta}_2 = -k_1(\tau)\zeta_2 + \zeta_4 - \frac{1}{2}(x_1+z_1)\zeta_3, \tag{A6b}$$

$$\dot{\zeta}_3 = \frac{1}{2}(x_1+z_1)\zeta_2, \tag{A6c}$$

$$\dot{\zeta}_4 = -a_2\zeta_2. \tag{A6d}$$

$$\dot{y}_1(\tau) = y_2(\tau) - y_1(\tau), \tag{A7a}$$

$$\dot{y}_2(\tau) = -\frac{1}{2}(x_1(\tau)+z_1(\tau))y_3(\tau) + y_4(\tau) - \frac{1}{2(C_1+C_2)}(x_1(\tau)-z_1(\tau))(x_3(\tau)-z_3(\tau)), \tag{A7b}$$

$$\dot{y}_3(\tau) = \frac{1}{2}(x_1(\tau)+z_1(\tau))y_2(\tau) - \frac{2}{C_1+C_2}a_1 + \frac{1}{2(C_1+C_2)}(x_1(\tau)-z_1(\tau))(x_2(\tau)-z_2(\tau)), \tag{A7c}$$

$$\dot{y}_4(\tau) = -a_2y_2(\tau). \tag{A7d}$$