

## Geometrical properties of coupled oscillators at synchronization

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### ABSTRACT

We study the synchronization of  $N$  nearest neighbors coupled oscillators in a ring. We derive an analytic form for the phase difference among neighboring oscillators which shows the dependency on the periodic boundary conditions. At synchronization, we find two distinct quantities which characterize four of the oscillators, two pairs of nearest neighbors, which are at the border of the clusters before total synchronization occurs. These oscillators are responsible for the saddle node bifurcation, of which only two of them have a phase-lock of phase difference equals  $\pm \pi/2$ . Using these properties we build a technique based on geometric properties and numerical observations to arrive to an exact analytic expression for the coupling strength at full synchronization and determine the two oscillators that have a phase-lock condition of  $\pm \pi/2$ .

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### 1. Introduction

Coupled oscillators have been used to understand the behavior of systems in physics, chemistry, biology, neurology as well as other disciplines. In particular, they are used to model phenomena such as: Josephson junction arrays, multimode lasers, vortex dynamics in fluids, biological information processes, neurodynamics [1–3]. These systems have been observed to synchronize themselves to a common frequency, when the coupling strength between the oscillators is increased [3–5]. Although all these phenomena have different dynamics, their synchronization features might be described using a simple model of weakly coupled phase oscillators such as the Kuramoto model and variations to adapt it for finite range interactions which are more realistic to represent physical systems [3–6]. But, finite range interactions difficult the analysis and search for analytical solutions. In spite of that, in order to figure out the collective phenomena when finite range interactions are considered, it is of fundamental importance to study and to understand the case of nearest neighbor interactions, which is the simplest form of the local interactions. In this context, a simplified version of the Kuramoto model with nearest neighbor coupling in a ring topology, which we shall refer to as locally coupled Kuramoto model (LCKM), is a good candidate to describe the behavior of coupled systems with local interactions. The LCKM has been used to represent the dynamics of a variety of systems. Specifically, it has been shown that a ladder array of Josephson junctions can be expressed by a LCKM [7]. Phase synchronization, in nearest neighbors coupled Rössler oscillators and locally coupled lasers where local interactions are dominant, can also be described by the LCKM [8–11]. Other examples are: the occurrence of travelling waves in neurons, chains in disorders, multicellular systems in biology, the dynamics of an edge dislocation in a 2D lattice and an antenna array in communication systems [3,4,6,12–14].

While in the Kuramoto model of long range interactions one has to get a solution in a mean field approximation, in the local model it is necessary to study the behavior of individual oscillators in order to understand the collective dynamics. Due

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to the difficulty in applying standard techniques of statistical mechanics in order to obtain a close picture of the effect of the local interactions on synchronization, we rely on a simple approach to understand the coupled system with local interactions, by means of numerical study of the temporal behavior of the individual oscillators. In this case, numerical investigations provide a good tool to understand the mechanism of interactions at the stage of complete synchronization which in turn helps to get an analytic solution. Earlier studies on the LCKM show several interesting features including tree structures with synchronized clusters, phase slips, bursting behavior, saddle node bifurcation and so on [15,16]. It has also been shown that neighboring elements share dominating frequencies in their time spectra, and that this feature plays an important role in the dynamics of formation of clusters in the local model [17,18]; that the order parameter, which measures the evolution of the phases of the nearest neighbor oscillators, becomes maximum at the partial synchronization points inside the tree of synchronization [19] and a scheme has been developed based on the method of Lagrange multipliers to estimate the critical coupling strength for complete synchronization in the local Kuramoto model with different boundary conditions [20]. In addition, based on numerical investigations, we identified two oscillators which are responsible for dragging the system into full synchronization [21], and the difference in phase for this pair is  $\pm \pi/2$ . These two oscillators are among two pairs of oscillators which are formed by the four oscillators at the borders between major clusters in the vicinity of the critical coupling. Using these findings we developed a method to obtain a mathematical expression for the estimated value of the critical coupling at which full synchronization occurs, once a set of initial conditions for the frequencies of the  $N$  oscillators is assigned [22].

In this work, we use the fact that at the stage of full synchronization, all oscillators have a common frequency and the time dependence of the phase difference among neighboring oscillators will vanish. Using this we are able to derive an analytic expression for the phases of oscillators, which shows the actual dependence on the periodic boundary condition. It is clear that this effect will decrease when the number of oscillators  $N$  increases, and the result will converge to that of a free chain as  $N$  tends to infinity [22]. Even so, both problems are different and have interesting peculiarities when the number of oscillators is finite. This problem is not a mere detail in a theoretical problem since a ring with a finite number of oscillators has many applications in electronics, coupled lasers and in communications [13,23,24]. In the process of finding the solution, using the saddle node bifurcation which dominates the dynamics at synchronization, we come across two quantities which will permit us to identify the four oscillators at the borders between major clusters, where only two of them will have a phase-lock condition of  $|\pi/2|$ . We use the properties of these oscillators and the ones at the boundaries to define a triangle and use its trigonometric properties to derive an analytic formula for the critical coupling. Finally, we can identify directly the pair of oscillators which has the phase-lock condition, which depends only on the set of initial frequencies.

This paper is organized as follows: In Section 2, we introduce the LCKM with boundary conditions. We determine analytically the critical coupling at the stage of complete synchronization. Finally, in Section 3 we give a conclusion which is based on a summary of the results.

## 2. The model

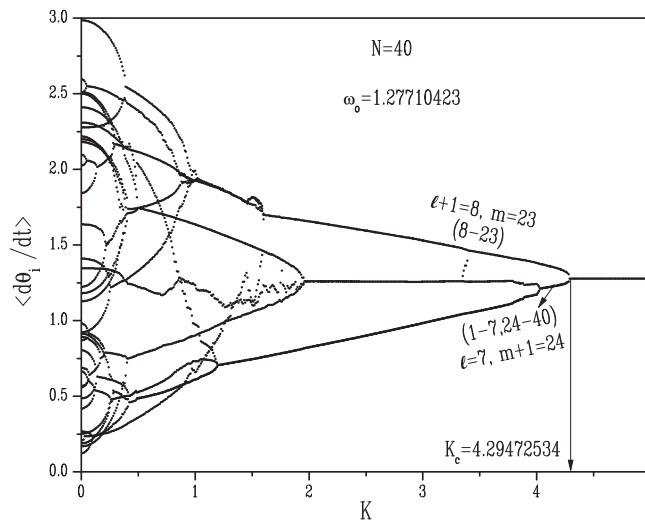
The LCKM can be considered as a diffusive version of the Kuramoto model, and it is expressed as

$$\dot{\theta}_i = \omega_i + K(\sin \phi_i - \sin \phi_{i-1}) \quad (1)$$

with periodic boundary conditions  $\theta_{i+N} = \theta_i$  and phase difference  $\phi_i = \theta_{i+1} - \theta_i$  for  $i = 1, 2, \dots, N$ . The set of the initial values of frequencies  $\omega_i$  are the natural frequencies which are taken from a Gaussian distribution and  $K$  is the coupling strength. These nonidentical oscillators of system 1 cluster in time averaged frequency, under the influence of the coupling, until they completely synchronize to a common value given by the average frequency  $\omega_0 = \frac{1}{N} \sum_{i=1}^N \omega_i$  at a critical coupling  $K_c$  as shown in Fig. 1. At the vicinity of  $K_c$ , there are only two clusters of successive oscillators whose borders are nearest neighbors. The oscillators remain synchronized for  $K \geq K_c$  when all phase differences and frequencies are constants. In Fig. 1, we show the synchronization tree for a system with  $N = 40$  oscillators, where the elements which compose each one of the major clusters, just before complete synchronization occurs, are indicated in each branch. These clusters merge into one at  $K_c$  where all oscillators have the same frequency. The major clusters, at the onset of synchronization, just before  $K_c$  contain  $N_1$  and  $N_2$  oscillators, where  $N = N_1 + N_2$ . It is not necessary for these clusters to have the same numbers of oscillators. We point to the oscillators at the border of each cluster, which are formed with successive elements, thus their bordering elements are nearest neighbors, as mentioned i.e.,  $\ell, \ell + 1, m$  and  $m + 1$ . An interesting fact emerges here: there is a phase-locked solution, where the phase difference between two oscillators is  $\pm \pi/2$ , and it is always valid for one and only one phase difference. This can be the difference between phases of any two nearest neighbors of the four oscillators at the border of the clusters at the onset of synchronization [21]. However, we can not directly allocate these two oscillators unless we do it numerically. At this point, we remind the reader that the location of each oscillator corresponds to a well defined entity, characterized by an initial frequency, therefore wondering about the meaning of the location of these borders as well as the position of the boundaries, corresponds to knowing which of the many oscillators in our systems will dominate the dynamics.

Thus, the time evolution of the phase differences among neighboring oscillators will be written as

$$\dot{\phi}_i = \Delta_i - 2K \sin(\phi_i) + K \sin(\phi_{i-1}) + K \sin(\phi_{i+1}), \quad (2)$$



**Fig. 1.** Synchronization tree for a system of 40 oscillators with detailed composition of each cluster before full synchronization. Here we point the oscillators at the border.

where  $\Delta_i = \omega_{i+1} - \omega_i$ . We use the fact that at  $K_c$ , the quantities  $\dot{\phi}_i = 0$ , to derive the following expression

$$\sin(\phi_i) = \frac{H_i}{K_c} + \sin(\phi_N), \tag{3}$$

where,  $H_i = \frac{(N-i+1)}{N} [\sum_{i=1}^{N-1} i\Delta_i + (\sum_{j=1}^{i-1} j\Delta_j)\delta_{ij}]$ . According to Eq. (3), we need to identify not only the two oscillators, say  $j$  and  $j + 1$ , which will phase-lock with  $\phi_j = |\pi/2|$ , but also the value of the phase difference between the oscillators identified as boundaries to find the critical coupling. In order to tackle this difficulty we depend on numerical simulations of system 1 for different number of oscillators and sets of initial frequencies  $\omega_i$ , as well as on studying the quantity  $H_i$ . It should be noted that the quantity  $\max\{|H_i|\}$  determines the value of the critical coupling and the oscillators locked on  $|\pi/2|$  at synchronization, for the case of a chain of free ends (open boundaries). This can be clearly understood from Eq. (3), when  $\sin(\phi_N) = 0$ , and  $|\sin(\phi_j)| = 1$ , which coincides with the same index of  $\max\{|H_i|\}$ , and henceforward  $K_c = \max\{|H_i|\}$ . However, when periodic boundary conditions are turned on, this fact does not hold true, since  $\sin(\phi_N)$  can be either positive or negative. Therefore, its presence in Eq. (3) modifies the quantity  $H_j$  which matches the same index of the phase difference which locks to  $|\pi/2|$ , and the oscillators no longer synchronize to a single frequency  $\omega_0$  at a critical coupling defined as  $\max\{|H_i|\}$ . Helped by numerical simulations, we can find the four oscillators at the edges of the two clusters at the onset of synchronization, in the vicinity of  $K_c$ , and see that only one phase difference is locked to  $|\pm \pi/2|$  at  $K_c$  as well as to determine which two oscillators among these four are responsible for the saddle node bifurcation.

When we study numerically the phase differences among neighboring oscillators we find two distinct values among all phase differences, which are characterized by being the absolute maximum and minimum among all phase differences and we shall see that they correspond to the two phase differences of the four oscillators at the borders [22]. But, only one of them will have the phase-lock condition of  $|\pm \pi/2|$  at  $K_c$ . If we now proceed to study the behavior of the quantity  $H_i$ , we see that the maximum and minimum values of  $H_i$  always coincide with the same indexes of the maximum and minimum of  $\sin(\phi_i)$ , respectively.

Therefore, based on the numerical simulations, we can distinguish two quantities  $H_\ell$  and  $H_m$  and their corresponding phase differences  $\phi_\ell$  and  $\phi_m$ . Any one of them,  $H_\ell$  or  $H_m$  can be the absolute maximum or the absolute minimum. This holds true for the two values  $\sin(\phi_\ell)$  and  $\sin(\phi_m)$  also. Comparing between the absolute values of either the two quantities  $H_\ell$  and  $H_m$  or the two quantities  $\sin(\phi_\ell)$  and  $\sin(\phi_m)$ , we can not say which oscillators will have the phase-lock condition. However, we find always, and for any  $N$  with any sets of  $\omega_i$ , that the signs of  $H_\ell$  and  $\sin(\phi_\ell)$ , as well as the corresponding  $H_m$  and  $\sin(\phi_m)$  are always the same, independent of being positive or negative. Fig. 2 shows the plot of selected values of  $\sin(\phi_i)$  versus time at  $K_c$  for the same realization of Fig. 1. We can see that  $\sin(\phi_\ell) = 1$  for  $\ell = 7$  and  $\sin(\phi_m) < 0$ , for  $m = 23$ , which is the absolute minimum among all corresponding values of all oscillators. Fig. 3 shows the values of  $H_i$  versus the positions of oscillators in the chain for the same case of Fig. 1, where we clearly see that  $H_\ell > 0$  for  $\ell = 7$  is the absolute maximum, while  $H_m < 0$  for  $m = 23$  is the absolute minimum.

Since these oscillators are obviously special, we shall focus on them in search for the exact solution of the problem. We see that the maximum and minimum of  $H_i$  determine two major cases:  $H_\ell > 0$  and  $H_m < 0$ . Once we select the first case, we notice that, again, we encounter two cases. We shall see that only one of them decides the exact value of the critical coupling. These two sub-cases are (a)  $\sin(\phi_\ell) = 1$  with  $\sin(\phi_m) > -1$ , and (b)  $\sin(\phi_\ell) < 1$  with  $\sin(\phi_m) = -1$ . For case (a) we use three equations from system (3) for  $\ell, \ell - 1$  and  $m$ , from which we can write  $\frac{\sin(\alpha/2)}{a} = \frac{\sin(\beta/2)}{b} = \frac{\sin(\gamma/2)}{c}$ . This relation corresponds to

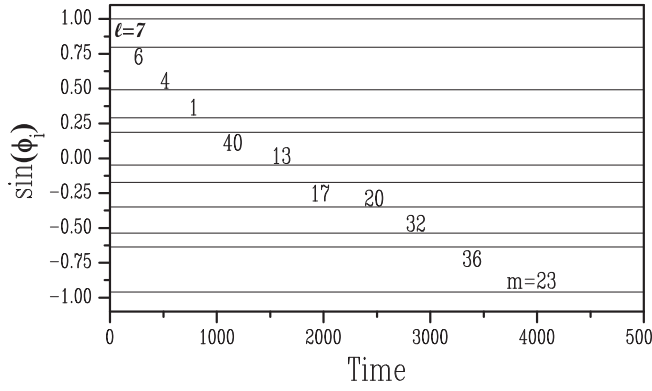


Fig. 2. Selected values of  $\sin(\phi_i)$  at  $K_c$  for the oscillators of Fig. 1.

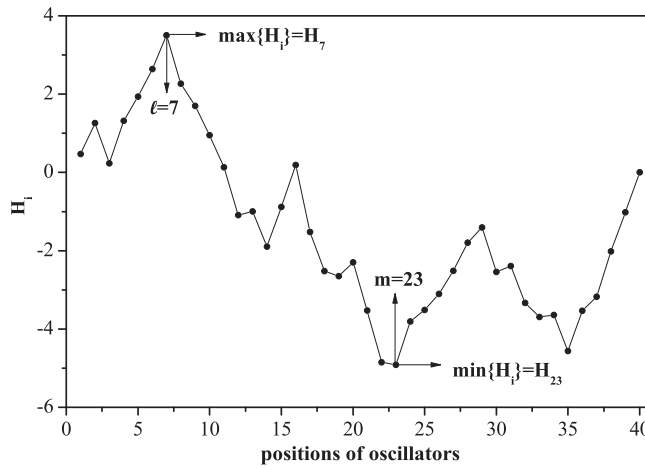


Fig. 3. The values of  $H_i$  versus oscillator position, for the oscillators of Fig. 1.

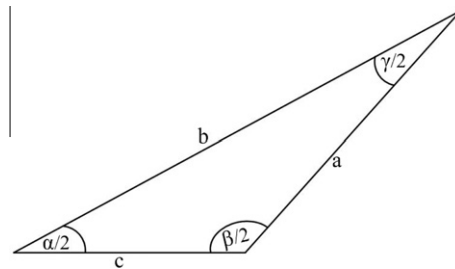


Fig. 4. Triangle of sides:  $a = (H_\ell)^{1/2}$ ,  $b = (H_\ell - H_m)^{1/2}$  and  $c = (H_\ell - H_{\ell-1})^{1/2}$ .

the properties between the angles and sides of a triangle, where the angles and sides are defined as:  $\alpha = -\phi_N + \pi/2$ ,  $\beta = -\phi_m + \pi/2$  and  $\gamma = -\phi_{\ell-1} + \pi/2$ ,  $a = (H_\ell)^{1/2}$ ,  $b = (H_\ell - H_m)^{1/2}$  and  $c = (H_\ell - H_{\ell-1})^{1/2}$ , as shown in Fig. 4. After some manipulation, we get an expression for  $\phi_m$  as

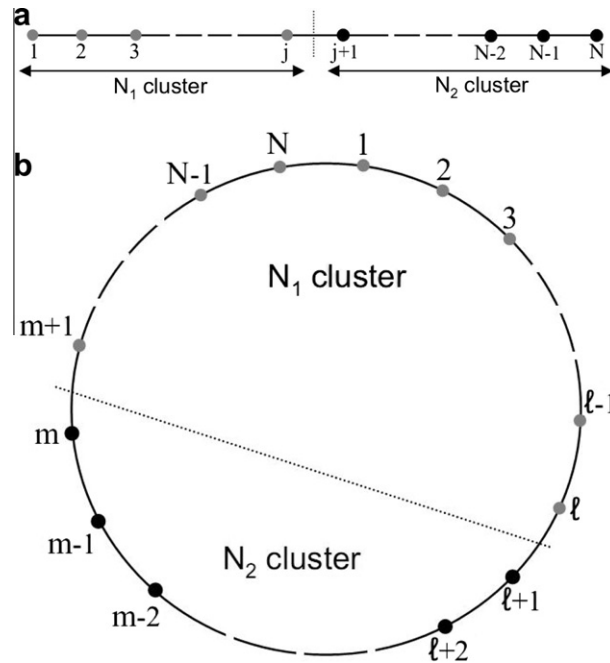
$$\phi_m = \frac{\pi}{2} - 2 \cos^{-1} \left\{ \frac{a^2 + c^2 - b^2}{2ac} \right\}. \tag{4}$$

Thus, the value of the critical coupling becomes:  $K_c^\ell = \frac{H_\ell - H_m}{1 - \sin(\phi_m)}$ . Applying the same method to case (b) ( $\sin(\phi_l) < 1$  with  $\sin(\phi_m) = -1$ ), we get  $K_c^m = \frac{-(H_m - H_\ell)}{1 + \sin(\phi_l)}$ . Therefore, the value of the critical coupling when  $H_\ell > 0$  and  $H_m < 0$  is

$$K_c = \max \{K_c^\ell, K_c^m\}. \tag{5}$$

**Table 1**Values of  $K_c$  from simulation of system 1 and from Eq. (5).

$N$	$K_c$ : simulation	$K_c$ : Eq. (5)
30	3.73094125	3.72862539
40	4.29473534	4.28553971
50	4.48415639	4.47935214
100	5.86827841	5.86639415
200	7.96802973	7.96457428

**Fig. 5.** The two major clusters at the critical coupling in (a) the chain of free ends and (b) the ring.

Following the same method for the case  $H_\ell < 0$  and  $H_m > 0$ , we find:  $K_c^\ell = \frac{-(H_\ell - H_m)}{1 + \sin(\phi_m)}$  and  $K_c^m = \frac{H_m - H_\ell}{1 - \sin(\phi_\ell)}$ . The value of the critical coupling is again given by Eq. (5). Table 1 shows the results from numerical simulations of system 1 which are in good agreement with the values obtained from Eq. (5) for the same sets of initial frequencies.

Eq. (5) allows us to determine whether  $\phi_\ell$  or  $\phi_m$  has the phase-lock condition  $|\pi/2|$  at  $K_c$ , depending whether the selected value for  $K_c$  is  $K_c^\ell$  or  $K_c^m$ , respectively. After this is done we can determine the value of  $\sin(\phi_N) \neq 0$ , thus showing how the phase difference  $\phi_N$ , consequence of the periodic boundary conditions, influences the dynamics of the system. We notice from Fig. 2 that the value of  $\sin(\phi_{40}) \neq 0$ , and it modifies the result of the critical coupling, as expected from Eq. (3). The numerical simulations, for different  $N$  and for different sets of  $\omega_i$ , show that the system with periodic boundary conditions has a critical value which depends on  $N$ , and that the influence of  $\sin(\phi_N)$  decreases as  $N$  increases which may give indications that it becomes ineffective, but in this case the system never synchronizes at the critical strength given only by the absolute maximum of  $H_i$ , as it should be for a free chain. We find also, that  $\sin(\phi_N)$  remains different from zero up to  $N \sim 2000$ . Even though,  $\sin(\phi_N)$  has a small value, but since it exists, it enforces the oscillators to synchronize approximately as an average of  $H_\ell$  and  $H_m$ , which are not in generally equal to each other. For larger values of  $N$ , we expect that  $|H_\ell| \approx |H_m|$  and  $\sin(\phi_N) = 0$ , and hence the critical coupling is determined by the absolute value of  $H_\ell$ . As  $N \rightarrow \infty$ , the system of oscillators in a ring synchronizes at the same value of the coupling strength as a chain of free ends, as expected [15].

Qualitatively, we can compare the cases of synchronization for a ring and for a free chain for finite number of oscillators in order to understand why the critical coupling depends on both values  $\max\{H_i\}$  and  $\min\{H_i\}$  for the first case while the free chain only depends on  $\max\{|H_i|\}$ . If we consider both systems above full synchronization and decrease the coupling strength we observe that a saddle node bifurcation occurs at the critical value and they split into two clusters of unequal number of oscillators in general. The chain splits into two clusters as shown in Fig. 5a, where only two oscillators seem to have a new role in the dynamics,  $j$  and  $j+1$  are located in different clusters. This splitting occurs when the phase difference  $\phi_j = |\pi/2|$ , and it is the index that maximizes  $\max\{|H_i|\}$  which determines the value of  $K_c$  for the chain. The ring at the critical coupling splits into two clusters of unequal numbers of oscillators, as shown in Fig. 5b. The difference is that for the ring two sets of

oscillators acquire a special role in the dynamics:  $\ell$ ,  $\ell + 1$ ,  $m$  and  $m + 1$  but only one phase difference will be  $\phi_j = |\pi/2|$  and it can be either  $\phi_\ell$  or  $\phi_m$ , where the two indexes  $\ell$  and  $m$  match  $\max\{H_i\}$  and  $\min\{H_i\}$ , respectively. A priori we do not know which will be. From the previous discussion we can write  $K_c = |H_j \pm \delta|_{\max}$ , where  $j = \ell$  or  $m$  and  $\delta$  depends on the periodic boundary conditions, in order to get  $|\sin(\phi_j)| = |X_j + \sin(\phi_N)| = 1$  and  $X_j = H_j/K_c \neq \pm 1$  and hence  $\sin(\phi_N)$  “decides” which will be the pair of oscillators which phases-lock at  $|\pi/2|$ . Therefore, for finite  $N$ , the oscillators synchronize at a critical coupling that depends on both quantities  $\max\{H_i\}$  and  $\min\{H_i\}$ , and this is due to the presence of the periodic boundary conditions. From Eqs. (3) and (5) we can write:  $K_c = \frac{H_\ell + |H_m|}{1 + |\sin(\phi_j)|}$ , with  $|\sin(\phi_j)| < 1$ , where  $j = m$  and  $|\sin(\phi_\ell)| = 1$ , and  $j = \ell$  when  $|\sin(\phi_m)| = 1$ .

### 3. Conclusions

We have studied the synchronization of coupled oscillators in a locally coupled Kuramoto model with nearest neighbors coupling with periodic boundary conditions. Particularly, we analyzed the system of oscillators at the stage of complete synchronization. We see that the dynamics is determined by a particular set of oscillators which are located at the borders of the major clusters which will meet at the critical coupling to form one cluster of all synchronized oscillators. Using the boundary conditions we can make a correspondence of the values of some definite quantities  $H_i$ 's with the sides of a triangle with the aid of some trigonometric properties. These quantities correspond to those of the border oscillators and their nearest neighbors. Using these two quantities, we derive a mathematical expression for the critical coupling at synchronization. In addition, we are also able to determine the two oscillators which will have a phase-lock condition of  $|\pi/2|$ . All these properties can be calculated a priori since they depend only on the set of initial frequencies  $\omega_i$ .

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