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Analytical calculation of the transition to complete phase synchronization in coupled oscillators

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Abstract. Here we present a system of coupled phase oscillators with nearest neighbors coupling, which we study for different boundary conditions. We concentrate at the transition to the total synchronization. We are able to develop exact solutions for the value of the coupling parameter when the system becomes completely synchronized, for the case of periodic boundary conditions as well as for a chain with fixed ends. We compare the results with those calculated numerically.

Keywords. Synchronization; coupled chaotic oscillators.

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1. Introduction

The fact that systems of coupled oscillators can describe problems in physics, chemistry, biology, neuroscience and many other disciplines, is already widely accepted by the scientific community. They have been used to model diverse phenomena such as Josephson junction arrays, multimode lasers, vortex dynamics in fluids, biological information processes and neurodynamics [1–8]. The coupled oscillators have been observed to synchronize themselves (say to a common frequency value) in a variety of ways, such as total, partial, generalized, lag, etc., when the coupling strength varies [9–16]. In spite of the diversity of dynamics, all these systems synchronize themselves to a common frequency, in a tree-like clustering behavior, when the coupling strength between these oscillators increases [17–23]. Among the coupled oscillator systems, the system of coupled limit cycles has attracted a lot of interest. The synchronization phenomena of such a system, in spite of the diversity of dynamics can be described using simple models of coupled phase equations.

Although there has been an extensive exploration of the dynamical behavior of the coupled limit cycles that show the synchronization phenomena, many interesting features remain unknown and little has been done analytically. Special attention has been given to the complex synchronization tree of a system of non-chaotic oscillators with nearest neighbors sinusoidal interactions, which becomes chaotic when the interaction is turned on [24–28]. For example, such a model is successfully used to describe the arrays of Josephson junctions in a ladder [25] and locally coupled laser arrays [7]. This system, which is a diffusive version of the Kuramoto model [4],

$$\dot{\theta}_i = \omega_i + \frac{K}{3}[\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)], \tag{1}$$

possesses all the features of phase synchronization of a system of chaotic oscillators in spite of its simplicity. Here ω_i are the natural frequencies, selected randomly from a normal Gaussian distribution, K is the coupling strength, θ_i is the instantaneous phase, $\dot{\theta}_i$ is the instantaneous frequency and $i = 1, 2, \ldots, N$. Such oscillators with nearest neighbors interaction have been seen in the literature to describe Josephson junctions, laser arrays and phase-locked loops [18,25]. These non-identical oscillators cluster in time averaged frequency, until they completely synchronize to a common value of the average frequency [24,26–29], $\langle\dot{\theta}_i\rangle = \langle\dot{\theta}_j\rangle$, $i \neq j$, where

$$\langle\dot{\theta}_i\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T \dot{\theta}_i(t)\mathrm{d}t. \tag{2}$$

Using periodic boundary conditions $\theta_{i+N} = \theta_i$, and scaling the frequencies such that

$$\frac{1}{N}\sum_{i=1}^{N} \omega_i = 0, \tag{3}$$

the above system (1) of N oscillators has a critical coupling strength $K = K_\mathrm{c}$, where at $K \geq K_\mathrm{c}$ a complete frequency synchronization can be observed and each θ_i is locked to a fixed value. There is no phase locking for $K < K_\mathrm{c}$ although the system has clusters of oscillators of the same *average* frequencies. Reducing further the coupling strength K, the number of units which cluster to the same average frequency decreases, increasing the number of clusters, i.e. the number of branches in the synchronization tree, until finally all oscillators are independent and acquire their natural frequencies.

The scaling condition in the case of system (1) limits the synchronization to zero frequency. When this scaling is not used [26,27,29], the general features of the system and the value of K_c will not change and the synchronization occurs via the same transition tree but shifted to a common frequency value

$$\omega_0 = \frac{1}{N}\sum_{i=1}^{N} \omega_i, \tag{4}$$

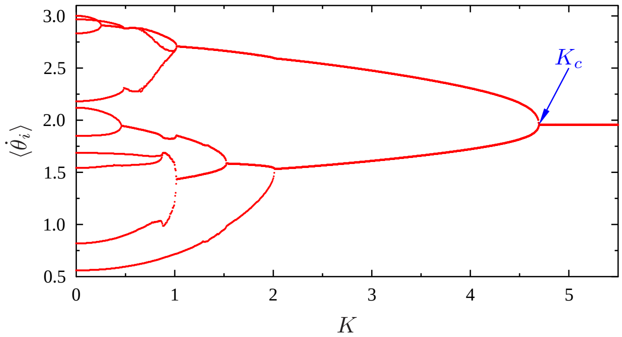

Figure 1. Synchronization tree for a system of $N = 10$ oscillators, with periodic boundary conditions ($K_{\rm c} \sim 4.70$).

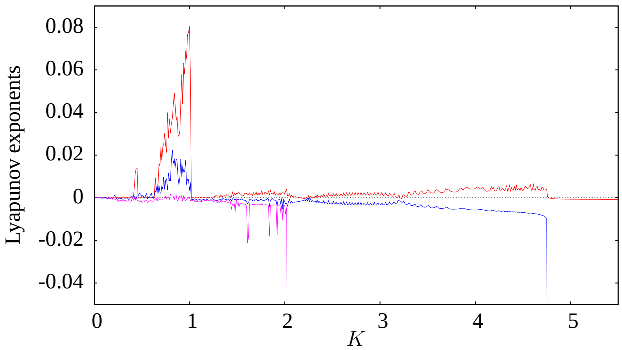

Figure 2. The first three largest Lyapunov exponents for a system with $N = 10$ oscillators.

as long as we maintain the periodic boundary conditions. When we lift these periodic boundary conditions, the system becomes dependent on the value of ω_0 [28], therefore, we shall keep ω_0 different from zero throughout this work.

In figure 1 we show the synchronization tree for a periodic system with $N = 10$ oscillators. In figure 2 we plot the first three largest Lyapunov exponents as a function of the coupling strength, K. At and above the critical coupling strength, $K_{\rm c}$, the largest Lyapunov exponent becomes zero with the rest of them being negative, an indication of synchronization. The initial frequencies for figures 1 and 2 are chosen as $\omega_i = \{0.5331, 2.8577, 2.6978, 2.0773, 2.8257, 1.6062, 0.7788, 1.4680, 1.7622, 2.0190\}$.

The analysis of synchronization and the loss of stability becomes particularly difficult since, as we mentioned above, it is on time average that these systems synchronize. Therefore, for system (1), one can perform analytic calculations only where the variables (phase and frequency) become time independent. In order to do that, it is necessary to make sure that we are in a zone of stationary states so that averages are equal to instantaneous values of the phases and frequencies. Henceforward, it becomes clear that we need to arrive at $K_{\rm c}$ from above ($K \geq K_{\rm c}$).

2. Method of Lagrange multipliers

2.1 *Periodic boundary conditions*

We shall start with the case of periodic boundary conditions. At complete synchronization, all oscillators have equal frequencies, which are time-independent. Since ω_0 is different from zero it is necessary to use the phase differences defined as $\phi_i = \theta_{i+1} - \theta_i$, which has been observed at the stage of synchronization to be time-independent as well as the sum of all phase differences are equal to zero. Then eq. (1) can be rewritten as

$$\dot{\phi}_i = \omega_i - \omega_{i-1} + \frac{K}{3}\left(\sin\phi_{i-1} - 2\sin\phi_i + \sin\phi_{i+1}\right). \tag{5}$$

In order to have complete frequency synchronization, the above systems of equations should have stable steady states. It can be seen from eq. (5) that the stability of the frequency steady states is independent of the initial frequencies as well as the coupling constant K provided $K > 0$. Thus, we could establish a criteria for the existence of stable steady states for a minimum value of $K > 0$ within the synchronization regime. Then simply the minimum value of K is equal to the critical coupling strength, K_c, since the system remains synchronized once full synchronization has been achieved (for $K \geq K_\mathrm{c}$). To find this minimum we use the method of Lagrange multipliers.

We should note that the frequencies of the system synchronize on time average, and after complete frequency synchronization all $\dot{\phi}_i = 0$, and the ϕ_i's are all constants. Therefore as long as the calculation is done above K_c, we can exchange the variables by their time-independent values. It is straightforward to see from the steady states of (5), where $\dot{\phi}_i = 0$, that

$$K = \frac{a_2}{\sin\phi_2^* - \sin\phi_1^*} = \frac{a_3}{\sin\phi_3^* - \sin\phi_1^*} = \cdots = \frac{a_N}{\sin\phi_N^* - \sin\phi_1^*} \tag{6}$$

and

$$\sin\phi_1^* - \sin\phi_2^* + \frac{a_3}{a_2}(\sin\phi_3^* - \sin\phi_1^*) = 0, \tag{7a}$$

$$\sin\phi_1^* - \sin\phi_2^* + \frac{a_4}{a_2}(\sin\phi_4^* - \sin\phi_1^*) = 0, \tag{7b}$$

$$\vdots$$

$$\sin\phi_1^* - \sin\phi_2^* + \frac{a_N}{a_2}(\sin\phi_N^* - \sin\phi_1^*) = 0, \tag{7c}$$

where

$$a_2 = 3(\omega_0 - \omega_1),$$

and

$$a_i = a_{i-1} + 3(\omega_0 - \omega_{i-1}), \quad i = 3, 4, \ldots, N. \tag{8}$$

In addition, from the definition of ϕ_i, it follows that

$$\sum_{i=1}^{N} \phi_i^* = 0. \tag{9}$$

We optimize the function (6) subject to the conditions (7) and (9). Then the minimum value of $K > 0$ is the critical value $K_{\rm c}$, provided that the set $\{\phi_i^*,\ i = 1, 2, \ldots, N\}$ form a stable steady state of (5).

For the above purpose, let us define a function $E(\phi_1^*, \phi_2^*, \ldots, \phi_N^*, \lambda_0, \lambda_3, \lambda_4, \ldots, \lambda_N)$ as

$$E = \frac{a_2}{\sin\phi_2^* - \sin\phi_1^*} + \sum_{i=3}^{N} \lambda_i \left[\sin\phi_1^* - \sin\phi_2^* + \frac{a_i}{a_2}(\sin\phi_i^* - \sin\phi_1^*)\right] + \lambda_0 \sum_{i=1}^{N} \phi_i^*, \tag{10}$$

where λ_i, $i = 3, 4, \ldots, N$ are parameters. To have an optimum K one should solve

$$\frac{\partial E}{\partial \phi_i^*} = 0, \quad i = 1, 2, 3, \ldots, N,$$
$$\frac{\partial E}{\partial \lambda_i} = 0, \quad i = 3, 4, \ldots, N. \tag{11}$$

The above conditions yield the following algebraic equation in addition to eqs (7) and (9):

$$\sum_{i=1}^{N} \frac{\cos\phi_1^*}{\cos\phi_i^*} = 0. \tag{12}$$

We can obtain the steady states ϕ_i^*, $i = 1, 2, \ldots, N$ values by solving eqs (7), (9) and (12). Hence we solve the above set of equations using Newton–Raphson method. Assuming a random set of initial values for ϕ_i^*'s, we look for converged and stable values for ϕ_i^*'s, then the values of $K > 0$ are estimated using eq. (6). The application of these equations bring the results shown in table 1. From table 1, we notice that the analytically and numerically calculated values are in a good agreement.

There exist in the literature some references to the solution of eqs (1) (see for example, refs [6,25,30]. But we need to notice that the difference of our case with theirs, is that we are looking at the time averaged solution of system (1). To solve our case we consider that in the above total synchronization ($K \geq K_{\rm c}$) all variables, phases and frequencies, are time-independent and the problem of handling temporal averages to estimate the value of the critical coupling $K_{\rm c}$, disappears. In addition, we are able to obtain a condition for complete synchronization to occur (eqs (7), (9) and (12)).

Table 1. Critical K values ($K_{\rm c}$) for the case of periodic boundary conditions.

	$K_{\rm c}$	
N	Numerical	Analytical
4	2.540775	2.540771
5	2.535587	2.535583
6	3.293280	3.293245
7	4.561624	4.561588
10	4.696059	4.696048
15	2.851727	2.851651
25	3.642530	3.642538
50	9.458391	9.458378
100	12.723401	12.723417

2.2 *Fixed ends*

We move now to the system where the phases at the boundaries are fixed. In this case, with the boundary conditions $\theta_N = c_N$ and $\theta_0 = c_0$, we can write the following equations:

$$\dot{\theta}_{N+1} = 0, \tag{13a}$$

$$\dot{\theta}_i = \omega_i + \frac{K}{3}[\sin(\theta_{i-1} - \theta_i) + \sin(\theta_{i+1} - \theta_i)], \tag{13b}$$

$$\dot{\theta}_0 = 0, \tag{13c}$$

where $i = 1, 2, \ldots, N$. Equation (13c) can be written in terms of the phase differences $\phi_i = \theta_i - \theta_{i-1}$ as

$$\begin{aligned}
\dot{\phi}_1 &= \omega_1 + \frac{K}{3}\left(\sin\phi_2 - \sin\phi_1\right), \\
\dot{\phi}_i &= \omega_i - \omega_{i-1} + \frac{K}{3}\left(\sin\phi_{i-1} - 2\sin\phi_i + \sin\phi_{i+1}\right), \\
\dot{\phi}_{N+1} &= -\omega_N - \frac{K}{3}\left(\sin\phi_{N+1} - \sin\phi_N\right),
\end{aligned} \tag{14}$$

where $i = 2, 3, \ldots, N$. Then, the constraint in this model reads as

$$\sum_{i=1}^{N+1} \phi_i = c_N - c_0. \tag{15}$$

The synchronization diagram is shown in figure 3 for $N = 10$ oscillators with initial frequencies $\omega_i = \{2.5331,\ 2.8577,\ 2.6978,\ 2.0773,\ 2.8257,\ 1.6062,\ 2.7788,\ 1.4680,\ 1.7622,\ 2.0190\}$ and the boundary conditions $c_0 = 0.15$ and $c_N = 2.15$. We notice that the oscillators synchronize with a frequency just below ω_0 at a coupling

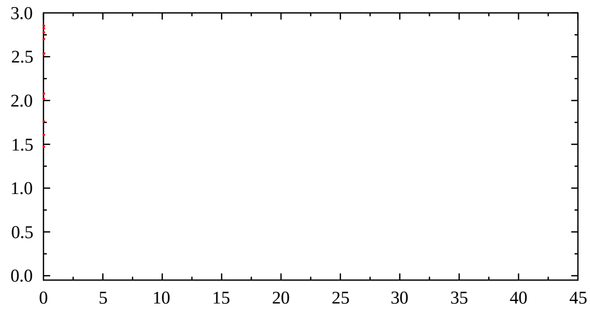

Figure 3. Synchronization tree for a system of $N = 10$ oscillators, with fixed ends ($K_s \sim 5.18$ and $K_\mathrm{c} \sim 41.13$).

Table 2. Critical K values (K_c) for the case of fixed ends.

	K_c	
N	Numerical	Analytical
3	30.343908	30.343892
6	76.993996	76.993767
10	41.127962	41.127924
12	68.656599	68.656533
15	53.059077	53.059054
25	53.968662	53.967955
50	115.546391	115.546343
100	80.218507	80.218133

constant K_s. Moreover, K_s depends on the initial frequencies and the values of the phases at the fixed ends, i.e. c_0 and c_N. Increasing the coupling constant, the oscillators remain synchronized in averaged frequency although this value of the average frequency decreases as K increases. Finally at $K = K_\mathrm{c}$, the average frequency collapses to zero. The value of K_c depends on ω_0 as well as on the boundary conditions in the phases of the oscillators [28].

From the steady states of eqs (13c) we find that

$$K = \frac{a_2}{\sin\phi_2^* - \sin\phi_1^*} = \frac{a_3}{\sin\phi_3^* - \sin\phi_1^*} = \cdots = \frac{a_N}{\sin\phi_N^* - \sin\phi_1^*} \tag{16}$$

and

$$\sin\phi_1^* - \sin\phi_2^* + \frac{a_3}{a_2}(\sin\phi_3^* - \sin\phi_1^*) = 0, \tag{17a}$$

$$\sin\phi_1^* - \sin\phi_2^* + \frac{a_4}{a_2}(\sin\phi_4^* - \sin\phi_1^*) = 0, \tag{17b}$$

$$\vdots$$

$$\sin\phi_1^* - \sin\phi_2^* + \frac{a_N}{a_2}(\sin\phi_N^* - \sin\phi_1^*) = 0, \tag{17c}$$

where

$$a_i = -3\sum_{j=1}^{i-1}\omega_j, \quad i = 2, 3, 4, \ldots, N. \tag{18}$$

The equation to be minimized by using Lagrange multipliers is

$$E = \frac{a_2}{\sin\phi_2^* - \sin\phi_1^*} - \sum_{i=3}^{N}\lambda_i\left[\sin\phi_1^* - \sin\phi_2^* + \frac{a_i}{a_2}(\sin\phi_i^* - \sin\phi_1^*)\right] + \lambda_0\left[\left(\sum_{i=1}^{N}\phi_i^*\right) - (c_N - c_0)\right]. \tag{19}$$

Upon minimizing (18), we obtain

$$\sum_{i=1}^{N}\frac{\cos\phi_1^*\cos\phi_2^*}{\cos\phi_i^*} = 0. \tag{20}$$

The solution of eqs (15), (17) and (20) gives K_{c} provided that we are at stable steady states ϕ_i^*, $i = 1, 2, 3, \ldots, N$. In table 2 we show the results obtained for the case of fixed ends, where it is easy to see that the analytic and numeric values of the critical coupling are in a good agreement.

3. Conclusions

In this paper we have performed analytic calculations, for a system of coupled oscillators through nearest-neighbor coupling with different boundary conditions, at the transition to complete synchronization. These oscillators are known to synchronize on time average frequencies, forming clusters, increasing the number of oscillators in each, until they come to complete synchronization. Until this is achieved the oscillators remain time-dependent, chaotic or otherwise, as shown by the Lyapunov exponent from figure 2. We calculated the value of the coupling strength, K_{c}, at complete synchronization. We found that the analytically and numerically calculated values are in a good agreement. In addition we obtain conditions for a complete synchronization to occur.

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