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# Controlling global stochasticity of Hamiltonian systems by nonlinear perturbation

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**Abstract.** A method of controlling global stochasticity in Hamiltonian systems by applying nonlinear perturbation is proposed. With the well-known standard map we demonstrate that this control method can convert global stochasticity into regular motion in a wide chaotic region for arbitrary initial condition, in which the control signal remains very weak after a few kicks. The system in which chaos has been controlled approximates to the original Hamiltonian system, and this approach appears robust against small external noise. The mechanism underlying this high control efficiency is intuitively explained.

**PACS.** 05.45.-a Nonlinear dynamics and non linear dynamical systems

In the recent decade, much attention has been paid to the control of chaotic dynamical systems using small perturbations [1–3]. There has been a strong focus on dissipative system control in the past. Due to the complicated transport properties, no great advancement has been achieved in controlling Hamiltonian chaos except some extensions of the OGY method [1,4] and chaotic targeting [5]. In particular, no work considered controlling global stochasticity, *i.e.*, global chaos, in two dimensional Hamiltonian systems, this is, however, of great importance in practical applications. For instance, in the tokamak problem which can be roughly reduced into a model of the standard map, we must be sure that the magnetic field lines (and, hopefully, the plasma) remain confined in the toroidal chamber [6].

In this work, we consider a nonlinear perturbation method to control global chaos in an example model of the standard map. In order to be possible of extending to handle quantum Hamiltonian system and to meet the physical conditions of most experiments, we expect the Hamiltonian system under control approximates to be conservative, at least when global stochasticity has been under controlled. The method we will discuss here does meet our expectation, and it is surprisingly efficient. We are able to entirely eliminate chaos in a wide stochastic region by very weak perturbation after a few adaptive kicks, then the fundamental property of the original Hamiltonian is kept while chaos is successfully controlled.

The standard map [7–9], which is one of the most extensively investigated models in many different applications, can be written in

$$\begin{aligned} J_{n+1} &= J_n - \frac{K}{2\pi}\sin(2\pi\theta_n) \\ \theta_{n+1} &= \theta_n + J_{n+1} \end{aligned} \tag{1}$$

it exhibits, in spite of its simple formulae, much of the complex and canonical behavior of more complicated realistic systems, and this is ideally suited for the study of chaotic dynamics in Hamiltonian systems [10]. For $K$ less than the threshold value $K_c \cong 0.9716...$, the motion in $J$ is bounded by the existence of good KAM surfaces [11,12]. For $K > K_c$, there is unbounded motion in $J$, *i.e.*, $J$ can be accelerated to infinity, and then global chaos sets in.

We are interested in the parameter range of $4 > K > K_c$, where chaotic orbits which can reach arbitrary values of $J$ while at least the origin $(0,0)$ is still elliptic, and a stable island around this point remains. Our task is to apply a weak nonlinear perturbation convenient in practice to destroy global stochasticity of the system, and to confine the system to low $J$ values, and convert chaos to regular motions. For this purpose we introduce the following control mechanism

$$\begin{aligned} J_{n+1} &= J_n - \frac{K}{2\pi}\sin(2\pi\theta_n) - \frac{e}{2\pi}\sinh(2\pi\theta_n) \\ \theta_{n+1} &= \theta_n + J_{n+1} \qquad \widehat{\text{mod}}\ 1 \end{aligned} \tag{2}$$

where the control signal $-\frac{e}{2\pi}\sinh(2\pi\theta_n)$ does not change the conservativity of the system. The notation $\widehat{\text{mod}}\ 1$

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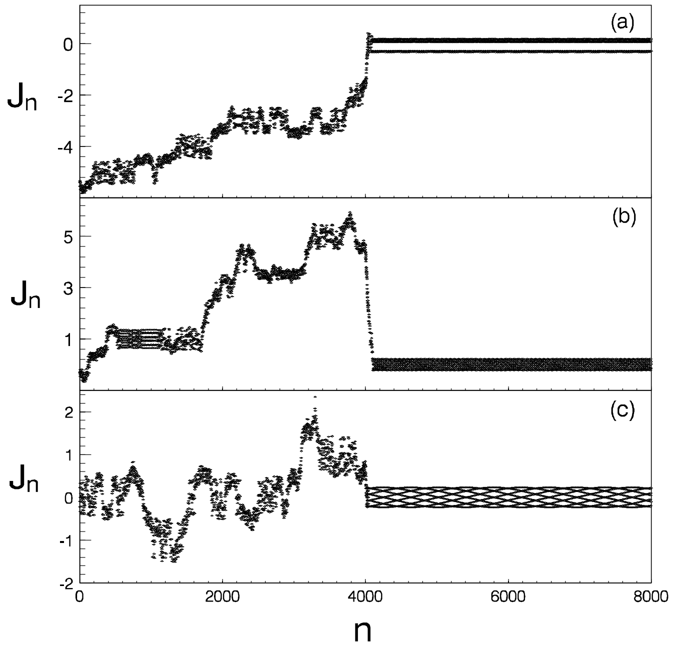


**Fig. 1.** The stabilized motions. $K = 1.5$, $e = 0.005$, and the control is switched on at $n = 4000$. (a), (b) and (c) have different initial values.

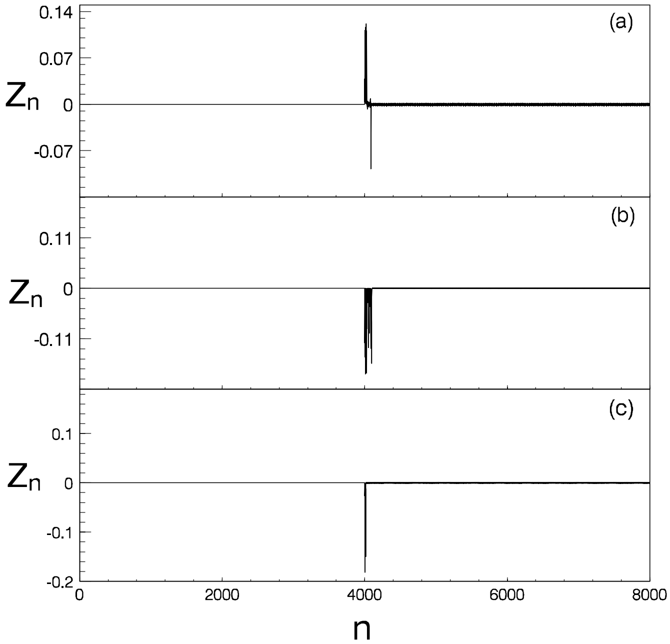


**Fig. 2.** The control signal $Z_n$ *vs.* $n$. (a–c) correspond to the cases of Figures 1a–c, respectively. $Z_n$ is described in equation (4).

means to take the fractional part of the variable in the $[-1, 1]$ region. Precisely, we take $\theta_{n+1} - n$ for either $0 < n < \theta_{n+1} < n + 1$ or $n - 1 < \theta_{n+1} < n < 0$. The function mod 1 introduces a discontinuity of the control signal $-\frac{e}{2\pi}\sinh(2\pi\theta_n)$ at $\theta_n = \pm 1$, this discontinuity will be shown to be crucial for the control efficiency of our approach. Equation (2) describes a practical Hamiltonian system in a periodic potential subject to a discontinuous control force, which varies continuously in the range $(-1, 1)$.

Numerical simulations show that global stochasticity can be effectively controlled into regular motions, and the final states are always quasiperiodic ones. Figure 1 presents three typical changes of the motions in $J_n$ when $K$ is fixed on 1.5. We have $e = 0$ for $n < 4000$, the control with $e = 0.005$ is applied at $n = 4000$. Figure 1a, b, c correspond to different initial conditions. The relaxation times for stabilizing the system are rather short, and this is of some importance in practice since long chaotic transient for chaos control may cause some serious damages in experimental systems. It is interesting to see that all kinds of orbits are finally controlled into regular motions in the neighborhood of the island close to the origin, which will be roughly called as the attractive basin in the following context. Let us consider the control signal

$$Z_n = -\frac{e}{2\pi}\sinh(2\pi\theta_n) \tag{3}$$

and plot $Z_n$ *versus* $n$ in Figures 2a, b, c, corresponding to the evolutions in Figure 1a, b and c, respectively. We note, in each case when the control is just switched on, $Z_n$ exhibits some large jumps in a few steps first, and these discontinuous forces drive the global stochasticity states of the original system into the attractive basin in the phase space. Then, the control signal keeps on an oscillation with an amplitude less than 0.001,which is incomparably smaller than the nonlinear force in the map $\left|\frac{K}{2\pi}\sin(2\pi\theta_n)\right|$. Therefore, once the chaotic system has been kicked into the attractive basin, the controlled system approximate to the original Hamiltonian system at the corresponding regular state. Referring to Figure 1, once $Z_n$ in Figure 2 exhibits stable oscillation, the original unbounded stochastic motion of the system can be constrained into a local regular quasiperiodic one. So that, this control method can be regarded as conservative control in the meaning of approximation. It deserves to note that keeping conservative is more close to the physical condition of many real systems and which enables extending the control idea into possible quantum systems as well.

Figure 3a shows some typical global chaotic orbits for $K = 1.5$ without control. Under the control of $e = 0.005$ they all turn into quasiperiodic motions in limited region around the origin in Figure 3b. This is true for all the range of $4 > K > K_c$. For instance, Figure 3c gives one example for $K = 3.5$ without control and Figure 3d control Figure 3c with $e = 0.008$. The striking point is that we can perfectly control global stochasticity of the system from arbitrary initial condition by applying a simple control signal of equation (2) with a very small control intensity. Our method is generally valid in the wide range of $K < 4$. We have checked $4 \times 10^4$ random initial values, all the motions are controlled into the typical orbits similar to those in Figure 1 and Figures 3b, d. There is no exception for the parameter range of $K_c < K < 4$, and their relaxation times are different from $10^1$ to $10^4$. Even for $K$ slightly larger than 4, some global chaotic orbits can still

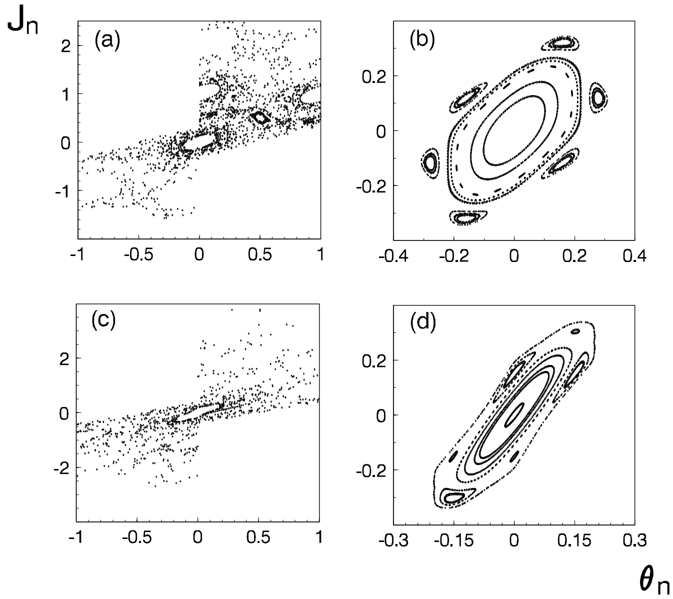


**Fig. 3.** (a) The typical global chaotic orbits of equation (1) with $K = 1.5$. (b) The typical solutions of equation (2) with $K = 1.5$ and $e = 0.005$. (c) The same as (a) with $K = 3.5$. (d) The same as (b) with $K = 3.5$ and $e = 0.008$.

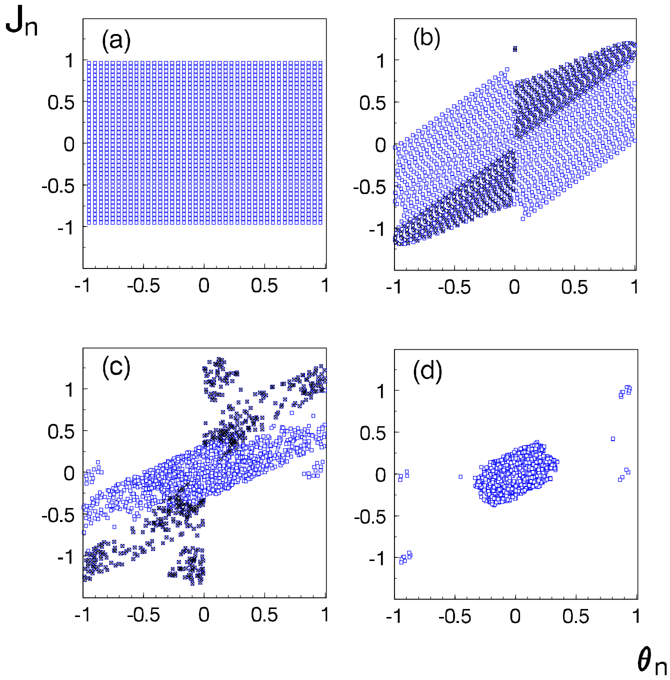


**Fig. 4.** $K = 1.5$, $e = 0.005$. Shaded regions with crosses in the figures are the discontinuous-map-induced overlapping regions. (a) Initial values distributed uniformly in the phase space. (b) The pattern after one iteration from (a). (c) The pattern after ten iterations from (a). (d) The pattern after 10000 iterations from (a).

be controlled into regular ones, and we had observed such example at $K = 4.3$ with $e = 0.01$.

Now it is of crucial importance to study the mechanism underlying the features of the control approach of Figures 1–3. The following points are emphasized.

(1) The most interesting feature of Figures 1–3 is that the violent global stochasticity can be effectively turned to localized regular motions where the control signals are extremely small. This feature is very much welcome because with small control the essential nature of the original system can be kept. Since the small control signal do not considerably change the stability property of the unperturbed system, the final state under control should be close to the stable and weakly unstable region of the uncontrolled system. This conclusion is confirmed by all Figures 2 and 3 where the control signal always drive the system to the island near $(0, 0)$. In this region small signals are enough to overcome the weak instability or to keep stability.

(2) Then the next crucial point is the mechanism how the control signal drives the system to the stable or weakly unstable region, the discontinuity of the control signal at $\theta_n = \pm 1$ is the key reason. Without control no any trajectory and any space region is attractive due to the Hamiltonian nature of the system. Actually, for a standard map, if the system starts from any space point it will soon or later comes back to the region infinitely close to the starting point on the premise of mod 1 to $(\theta_n, J_n)$. Therefore, no migration of the system from any region to the stable region can be expected without control. The control signal and $\widehat{\text{mod}}$ 1 projection change the invertible system to non-invertible, that makes the appearance of an "attractive" region. In Figure 4 we start from the region $J_0 = [-1, 1]$, $\theta_0 = [-1, 1]$ and iterate the system. After some iterations the initial area shrinks to a small asymptotic area centered at $(0, 0)$ due to the overlap of the iterated region. Occasionally (or by intelligent selection of the control signal), this asymptotic "attractive" region is just in the stable or weakly unstable domain of the uncontrolled system, then a successful control with weak signal can be surely anticipated, this is what happens in Figures 1–3.

(3) The existence of "attractive region" in a discontinuous conservative map shows dissipative-like behavior in some sense, and then called quasi-dissipativity [13]. Nevertheless, the conservative nature is kept in our control system. In particular, when the system is well under control, no discontinuity can be felt in the asymptotic motion, then the controlled system is typical Hamiltonian system with continuous force. It is emphasized that no any trajectory is attractive even under control, therefore, infinitely many final regular states can be realized from different initial states.

(4) One more question important for applications is whether the discontinuous control in equation (2) can be easily realized in experiments. The answer is positive. The control signal in (2) is a typical threshold system. The signal shows well behaved continuous function when a variable is below a threshold and discontinuously drop down (by discharge in electric circuits, for instance) after the variable is over the threshold. This kind of devices can be

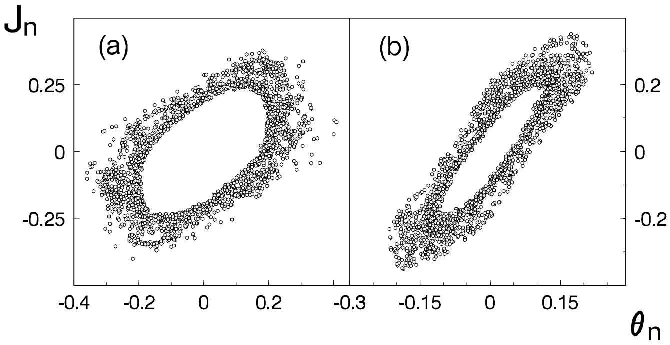


**Fig. 5.** Typical system orbits under both control and Gaussian white noises, equations (5). (a) $K = 1.5$, $e = 0.005$, and $\rho = 1.4\times10^{-4}$, corresponding to Figure 3b without noise. (b) $K = 3.5$, $e = 0.008$, and $\rho = 2.8\times10^{-5}$, corresponding to Figure 3d without noise.

easily designed for electric circuits [13], and is expected to realizable for plasma systems.

For applicability it is important to further exam the robustness of our control strategy, *e.g.*, we need to consider the influence of noise on the control stability. We consider the control system subject to noise

$$\begin{aligned} J_{n+1} &= J_n - \frac{K}{2\pi}\sin(2\pi\theta_n) - \frac{e}{2\pi}\sinh(2\pi\theta_n) + \rho\xi_n \\ \theta_{n+1} &= \theta_n + J_{n+1} + \rho\eta_n \qquad \widehat{\mathrm{mod}}\, 1 \\ \langle\xi_n\xi_{n'}\rangle &= \langle\eta_n\eta_{n'}\rangle = \delta(n-n'), \\ \langle\xi_n\eta_{n'}\rangle &= 0, \quad \langle\xi_n\rangle = \langle\eta_n\rangle = 0 \end{aligned} \tag{4}$$

where $\xi_n$ and $\eta_n$ are Gaussian white noise generated by using the Box-Müller method [14], and $\rho$ denotes the intensity of external noises.

Figure 5 shows the results obtained at the presence of noise for $K = 1.5$ in Figure 3b and $K = 3.5$ in Figure 3d, the noise intensity in Figure 5a and b is $1.4\times10^{-4}$ and $2.8\times10^{-5}$, respectively. Comparing Figure 5a with Figure 3b and Figure 5b with Figure 3d, we can see that noise wipes the fine structure of the orbits, however, the control of global stochasticity is not affected, *i.e.*, the trajectories are confined within a small neighborhood of the noise-free orbit and do not wander over the phase space to infinity. Therefore, the stability of weak nonlinear perturbation control is robust against weak external noise.

In summary, we proposed a weak nonlinear perturbation method to control global stochasticity in two-dimensional Hamiltonian systems, and we demonstrated its validity, high efficiency, and robustness in a model of the standard map. Though we take a particular model as our example, the following idea works generally. In most of chaotic conservative systems there exist some small regions allowing stable or weakly unstable motion. Without control for major portion of initial conditions the system cannot enter these regions and the motion is violently chaotic. Intelligently chosen non-invertible control signals may migrate the system to this favorable region from arbitrary initial conditions, and then eliminate chaos. Since the control signal approximates zero after a number of discontinuous kicks, the controlled system approximates to the original Hamiltonian system quickly, and this is favored to meet the possible practical requirements.

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## References


1. E. Ott, C. Grebogi, J.A. Yorke, Phys. Rev. Lett. **64**, 1996 (1990)
2. K. Pyragas, Phys. Lett. A **170**, 421 (1992)
3. T. Shinbrot, C. Grebogi, E. Ott, J.A. Yorke, Nature (London) **363**, 411 (1993); T. Kapitaniak, Controlling Chaos (Academic, London, 1995), M. Ding, E.-J. Ding, W.L. Ditto, B. Gluckman, V. In, J.-H. Peng, M. Spano, W. Yang, Chaos **7**, 644 (1997) T.F. Qin, Commun. Theor. Phys. **30**, 187 (1998); J. Xiao, G. Hu, J. Yang, J. Gao, Phys. Rev. Lett. **81**, 5552 (1998)
4. Y.C. Lai, M. Ding, C. Grebogi, J.A. Yorke, Phys. Rev. E **47**, 86 (1993)
5. C.G. Schroer, E. Ott, Chaos **7**, 512 (1997)
6. R. Balescu, Phys. Rev. E **55**, 2465 (1997)
7. A.J. Lichtenberg, M.A. Liebermann, Regular and Stochastic Motion (Springer, New York, 1983)
8. L.E. Reichl, *The Transition to Chaos* (Springer, New York, 1992); J.D. Meiss, Rev. Mod. Phys. **64**, 795 (1992)
9. B.V. Chirikov, Phys. Rep. **52**, 264 (1979)
10. R.B. White, S. Benkada, S. Kassibrakis, G.M. Zaslavsky, Chaos **8**, 757 (1998)
11. A.N. Kolmogorov, Dokl. Akad. Nauk SSSR **98**, 527 (1954); V.I. Arnold, Russian Math. Surveys **18**, 85 (1964); J. Moser, Nachr. Akad. Wiss. Goett. II, Math. -Phys. Kl. **1**, 1 (1962)
12. J.M. Greene, J. Math. Phys. **20**, 1183 (1979)
13. J. Wang, X.L. Ding, B. Hu, B.H. Wang, J.S. Mao, D.R. He, Phys. Rev. E **64**, 026202 (2001); J. Wang, X.L. Ding, B.H. Wang, D.R. He, Chinese Phys. Lett. **18**, 13 (2001)
14. W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, New York, 1986)