Coherent-ordered transition in chaotic globally coupled maps

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A spatial coherent and temporally chaotic state in globally coupled maps exists in the strong coupling regime. After the coherence loses stability the whole system is attracted to a two-cluster attractor $[M_1,M_2]$. The number of elements in the clusters depends on the initial conditions, which are chosen at random. We find, numerically, that the number of elements in the clusters obeys a power law decay near the onset of the transition. The difference of the two clusters displays a temporal behavior characteristic of on-off intermittency, although the distribution of the laminar phases shows a phase transition as a function of its length, making it essentially different from the latter. [S1063-651X(96)07809-9]

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The transition routes to chaos in low-dimensional nonlinear dynamical systems have been well understood. One of them, the intermittency route, was classified into three types by Pomeau and Manneville [1]. The essential feature of intermittency is that a simple periodic orbit is replaced by a chaotic attractor, where the chaotic behavior is randomly interspersed with periodic behavior resembling that before the transition, in an intermittent fashion. Recently, the statistical distribution of a different type of intermittency in some low-dimensional nonlinear dynamical systems, called ''on-off'' intermittency, has been obtained analytically [2–10]. This intermittency is characterized by a two-state nature. The ''off'' state, which is nearly constant, and remains so for very long periods of time, is suddenly changed by random bursts, the ''on'' state, which departs quickly from and returns quickly to the ''off'' state. A self-organized on-off spatiotemporal intermittency has also been reported in a system of coupled maps via nearest-neighbors interaction [11]. In this paper we focus our attention on some transitions that take place in globally coupled chaotic systems.

Globally coupled systems are ubiquitous in nature. They arise naturally in studies of Josephson junction arrays, multimode laser, charge-density wave, oscillatory neuronal system, and so on [12–14]. As one of the simplest globally coupled system, the globally coupled map (GCM) has been the subject of intensive research in recent years. Some rather surprising and novel results, such as clustering, splay state, collective chaotic behavior, and violation of the law of large numbers in the turbulent regime are revealed in the GCM model [15–18]. In this paper we will study the transition to intermittency in the GCM model, which takes place between the coherent and the ordered phases.

Specifically, we use the following form of GCM:

$$x_{n+1}^i=(1-\epsilon)f(x_n^i)+\frac{\epsilon}{N}\sum_{j=1}^{N} f(x_n^j), \quad i=1,2,\ldots,N, \quad (1)$$

where n, i, and ϵ are the discrete time step, the index of elements, and the coupling coefficient, respectively. The mapping function $f(x)$ is taken as the logistic map $f(x)=ax(1-x)$, and a is the nonlinear parameter. N is the total number of elements or system size.

An important concept in GCM model is ''clustering.'' This means that even when the interactions between all elements are identical, the dynamics can break into different clusters, each of which consists of fully synchronized elements. After the system falls in an attractor, we say that the elements i and j belong to the same cluster if $x_n^i \equiv x_n^j$. Therefore, the behavior of the whole system can be characterized by the number of clusters $n_{\rm cl}$, and the number of elements of each cluster $(M_1,M_2,\ldots,M_{n_{cl}})$ [15].

As the nonlinearity or coupling strength is varied, the system exhibits successive phase transitions among coherent, order, and turbulent phases [15]. We shall study the transition from the coherent chaotic state to a two-cluster chaotic attractor in the strong coupling regime.

In the coherent chaotic region the system is homogeneous in space, i.e., $x^i \equiv x^j$, $\forall i,j$, and chaotic in time. Thus, it is characterized by only one cluster, i.e., $n_1=1$, $M_1=N$. The motion of each element is equivalent to that of the single logistic map. The stability condition for this coherent state is that modulus of all eigenvalues of the $N\times N$ stability matrix $J=\prod_{n=1}^{m} f'(x_n) J_0{}^m$ has magnitude less than *one*. Here $f'(x_n)$ is the derivative of the nth iteration of the logistic map; m is taken as the periodic number or infinity for periodic or chaotic motions, respectively. $\mathbf{J_0}$ is a $N\times N$ constant matrix given by

$$\begin{pmatrix} 1-\epsilon+\frac{\epsilon}{N} & \frac{\epsilon}{N} & \cdots & \frac{\epsilon}{N} \\ \frac{\epsilon}{N} & 1-\epsilon+\frac{\epsilon}{N} & \cdots & \frac{\epsilon}{N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\epsilon}{N} & \frac{\epsilon}{N} & \cdots & 1-\epsilon+\frac{\epsilon}{N} \end{pmatrix}. \quad (2)$$

The matrix $\mathbf{J_0}$ is a circulant matrix, and can be written as

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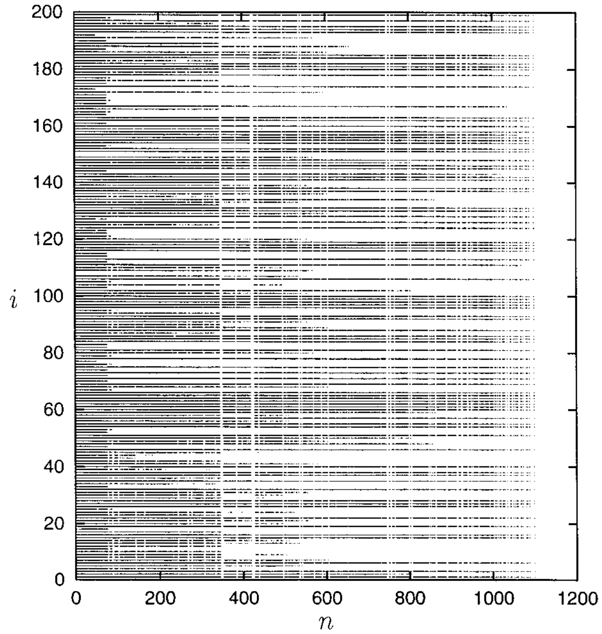

FIG. 1. Space-time evolution of the system at $N=200$, $a=4$, and $\epsilon=0.501$.

$$J_0=M_{\text{circ}}\left(1-\epsilon+\frac{\epsilon}{N},\frac{\epsilon}{N},\ldots,\frac{\epsilon}{N},\frac{\epsilon}{N}\right). \quad (3)$$

The eigenvalues of $\mathbf{J_0}$ are given by

$$\mu_{0,1}=1,$$

$$\mu_{0,r}=1-\epsilon+\frac{\epsilon}{N}\sum_{j=0}^{N-1} e^{i2\pi(r-1)j/N}\equiv 1-\epsilon,$$

$$r=2,3,\ldots,N. \quad (4)$$

Thus the eigenvalues of the stability matrix $\mathbf{J}$ are

$$\mu_1=\prod_{n=1}^{m} f'(x_n),$$

$$\mu_r\equiv(1-\epsilon)^m\prod_{n=1}^{m} f'(x_n), \quad r=2,3,\ldots,N. \quad (5)$$

The eigenvector corresponding to the eigenvalue μ_1 is given by $(1/\sqrt{N})(1,1,\ldots,1)^T$. Thus, the amplification of a disturbance along this eigenvector does not destroy the coherence. Eigenvectors for the other $N-1$ identical eigenvalues are not uniform; the amplification along these eigenvectors destroys the coherent phase. Therefore, the stability condition of the coherent chaotic state is decided by the $N-1$ identical eigenvalues. Their corresponding Lyapunov exponents are

$$\lambda=\lambda_r\equiv\ln(1-\epsilon)+\lambda_0, \quad r=2,3,\ldots,N, \quad (6)$$

where λ_0 is the Lyapunov exponent of the single logistic map. Therefore, the critical stability condition is given by $\lambda=0$, i.e., $\epsilon_c=1-e^{-\lambda_0}$. When ϵ is larger than ϵ_c, all elements quickly evolve to the same motion (the homogeneous state) after a short transient process, since $\lambda<0$. Generally speaking, we are only interested in the parameters where the behavior of the single logistic map is chaotic, $\lambda_0>0$. We have performed calculations for different values of a within the chaotic region. The results that we shall describe hold for all of them, therefore, we fix $a=4$ and $N=200$, where $\lambda_0=\ln 2$, thus, we have $\epsilon_c=\frac{1}{2}$. Figure 1 shows a space-time evolution at $\epsilon=0.501$. The initial condition of each element is randomly chosen in the uniform interval [0,1] throughout this paper. It is very clearly observed that all elements synchronize after almost 1100 iterations. We make the figure according to the rule: if $|x_n(i)-x_n(1)|>10^{-4}$, then the corresponding pixel is black, otherwise, it stays white.

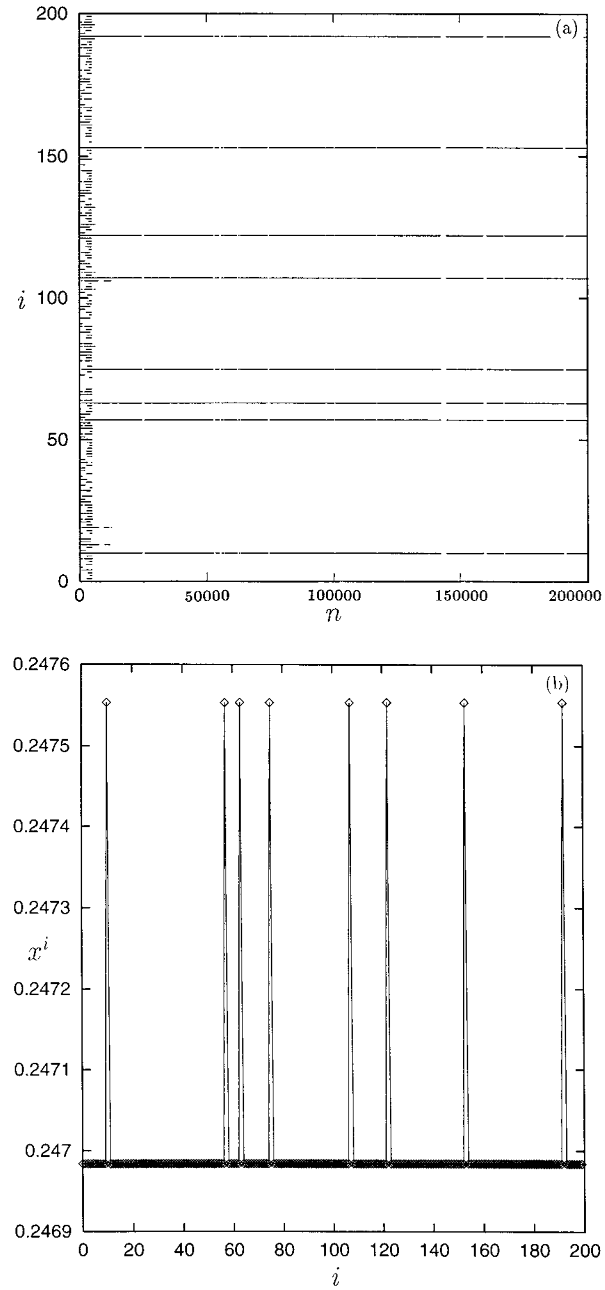

FIG. 2. (a) The same as in Fig. 1 for $\epsilon=0.499$. (b) The space structure of the system after the (3×10^5)th iterations of Eq. (1). Two clusters are clearly observed.

When the coupling ϵ is slightly smaller than the critical

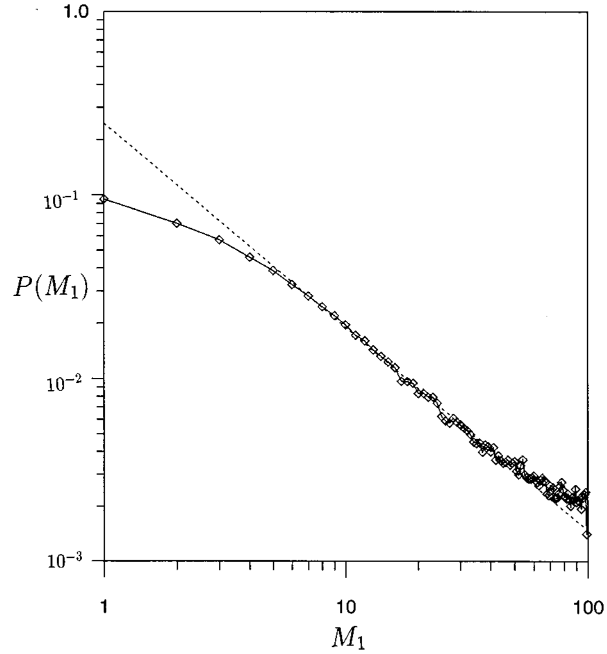

FIG. 3. The distribution of various M_1 (log-log plottings) for $\epsilon=0.499$. The dashed line is the perfect -1.11 power law decay.

value (0.5), the system suddenly evolves to a two-cluster attractor (M_1,M_2) $[M_1+M_2=N=200]$ after the transient process. Figure 2(a) shows a space-time evolution at $\epsilon=0.499$. After some iterations, the system is exactly set down to a two-cluster chaotic attractor $(M_1,M_2)=(8,192)$. The space structure after the (3×10^5)th iterations is also displayed in Fig. 2(b). Two clusters are clearly observed. M_1 and M_2 depend on the chosen random initial conditions. Since the sum of $M_1$and M_2 is always the same (200), only one of the two numbers can be varied freely. Assuming the

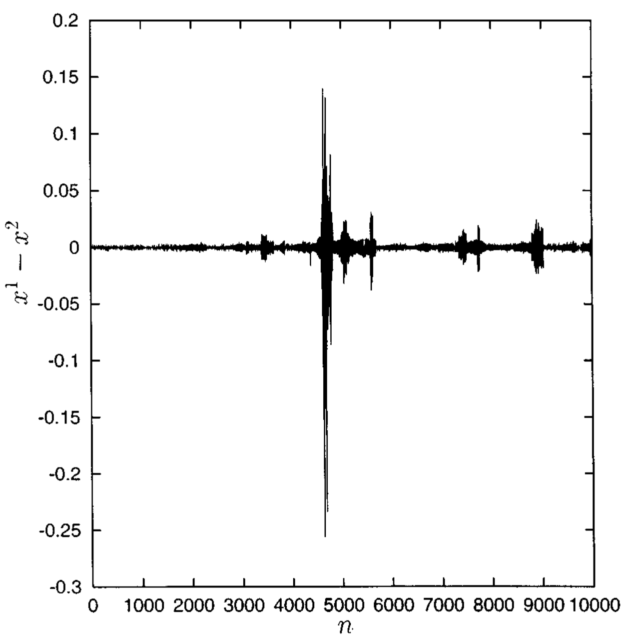

FIG. 4. The evolution of the difference of the two clusters after some transient process at $\epsilon=0.499$.

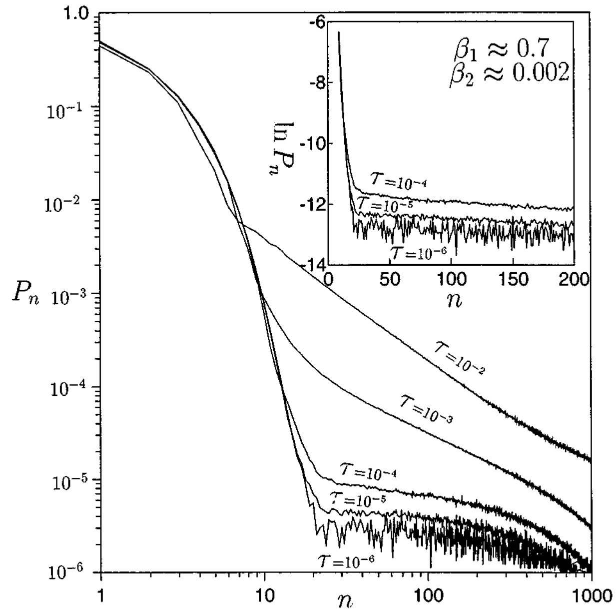

FIG. 5. The relative distribution P_n of the laminar phases of x^1-x^2 plotted against n (log-log plot in the large frame and n-log plot in the upper small frame) for several thresholds at $\epsilon=0.499$.

free variable to be M_1, with $M_1<M_2$, then we have $1\leqslant M_1\leqslant100$. We choose many random initial conditions, and iterate Eq. (1) for each one, then found, numerically, that the distribution of various M_1's obeys an exact power law decay as the control parameter crosses the critical value from above. Figure 3 shows the distribution of M_1 for $\epsilon=0.499$. A total of 10^5 different random initial conditions to iterate Eq. (1) were computed to obtain this curve. Except for the first several points, this distribution is a bonafide power law decay with an approximate exponent -1.11.

When the system falls in a two-cluster attractor, the dynamics can be replaced by

$$x_{n+1}^i=(1-\epsilon)f(x_n^i)+\frac{\epsilon}{N}\sum_{j=1}^{2} M_jf(x_n^j),\quad i=1,2. \tag{7}$$

Although the behavior of each cluster is chaotic, the difference of the two clusters (x^1-x^2) shows some very interesting and complex features. Figure 4 shows a time evolution of x^1-x^2 for the same parameter as those of Fig. 2. It is easily observed that x^1-x^2 remains a long time near zero, and suddenly departs from it and quickly returns after some random bursts. As the deviation from ϵ_c becomes large, more and more random bursts frequently occur.

In order to better characterize the intermittent behavior, we have calculated numerically the statistical distribution of the duration of the laminar phase x^1-x^2 shown in Fig. 5 for several thresholds of the difference τ at $\epsilon=0.499$. These thresholds for the laminar phase are defined by $|x^1-x^2|<\tau$, with τ ranging from 10^{-2} to 10^{-6}. For each threshold a total of 2×10^9 iterations of Eq. (1) were computed to obtain these curves. P_n represents the probability of the laminar phase of length n, namely, $P_n=M_n/M$, where M is the total number of segments of the laminar phase, M_n the number of

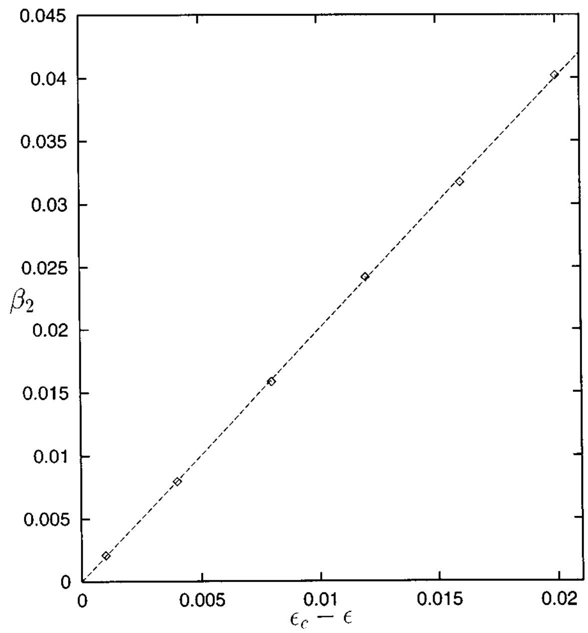

FIG. 6. β_2 vs $\epsilon_c-\epsilon$ at $\epsilon_c=0.5$. The diamonds are numerical results. The slope of the dashed line is 2.

those of length n. The distribution has the following remarkable property. For small segments of the laminar phase ($n<15$) the distribution quickly tends to the same exponential law decay with an asymptotic exponent -0.7 as the threshold τ decreases (see Fig. 5). We found numerically that this exponent is independent of the control parameters for small values of n, while for large segments of the laminar phase the distribution seems to depend on the threshold of the laminar phase. For large values of τ it obeys an asymptotic power law decay (see $\tau=10^{-2}$). As the threshold decreases, this power law is gradually replaced by another one. The new exponent ($-\beta_2$) depends on the deviation of the parameter ϵ from the critical value ϵ_c. Figure 6 shows the relation of β_2 and the coupling deviation $\epsilon_c-\epsilon$. It can be best fitted by

$$\beta_2=2(\epsilon_c-\epsilon). \tag{8}$$

Actually, this statistical distribution should be independent of the threshold chosen for the laminar phase. In order to get the invariant distribution, we take smaller values of τ (10^{-5} is enough). The invariant distribution can be approximately formulated as

$$P_n \propto \begin{cases} e^{-\beta_1 n}, & n<n_s, \\ e^{-2(\epsilon_c-\epsilon)n}, & n>n_s, \end{cases} \tag{9}$$

where $\beta_1\approx 0.7$ and $n_s\approx 15$. Thus, the distribution of the laminar phase described in this work shows a transition at n_s. Although this type of intermittency has similar characteristics to those of the conventional on-off intermittency (see Fig. 4), this transition does not exist in the latter, since its distribution obeys an asymptotic power law near the onset, with exponent $-\frac{3}{2}$.

In conclusion we have investigated the intermittency transition from a coherent chaotic state to a two-cluster chaotic attractor in globally coupled systems. We have found a new intermittency transition for globally coupled maps. We found that the numbers of elements in the clusters obey a power law decay near the onset of the transition. We have seen that this type of intermittency is essentially different from the types of intermittency known before, showing a phase transition as a function of the length of the laminar phase. The features of this type of intermittency are rather generic for globally coupled chaotic systems and are independent of the local mapping function, which we have taken as the logistic map to illustrate the phenomenon. Both spatially global uniform coupling and chaotic motion of the individual elements are of crucial importance.

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