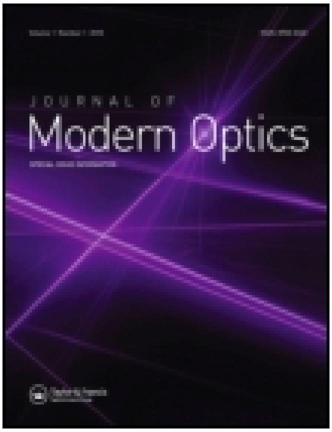

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Lifetimes of Chaotic Attractors in a Multidimensional Laser System

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Lifetimes of chaotic attractors in a multidimensional laser system

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Abstract. The lifetimes of chaotic attractors at crises in a multidimensional laser system are studied. This system describes the CO_2 laser with modulated losses and is known as the four-level model. The critical exponents which are related to the lifetimes of the attractors are estimated in terms of the corresponding eigenvalues and the measured characteristic lifetimes in the model. The critical exponents in this model and those of its centre manifold version are in good agreement. It is conjectured that generically in the four-level model the critical exponents are close to $\frac{1}{2}$ at crises. In addition, predictions of the simpler and popular two-level model are compared with those of the above-mentioned models

1. Introduction

Several studies in the last few years have been devoted to the theoretical, numerical and experimental analysis of the power-law scaling for the characteristic times near crises [1–4]. Crises are sudden changes that occur in a strange attractor as a control parameter is changed [1]. There are three different types of crises classified as attractor merging, attractor widening and attractor destruction. The latter applies to the sudden appearance of a new attractor, while the first two are characterized by intermittencies between different regions of the phase space [1]. Firstly, in this article we study the critical exponents in a multidimensional dynamical system known as the four-level model (4LM) which describes the dynamics of the CO_2 laser with modulated losses [5–7]. We underline that as regards the physics, the critical exponents are related to the mean lifetime of chaotic attractors in the vicinity of crises. Secondly, we find that a popular model for the CO_2 laser, known as the two-level model (2LM) has unrealistic eigenvalues for its unstable periodic orbits when compared to those extracted from the 4LM. The lifetimes of chaotic attractors as well as the eigenvalues of the unstable periodic orbits can be measured experimentally [2, 3].

A general theory to determine the critical exponents η for homoclinic and heteroclinic crises in three-dimensional (3D) flows or two-dimensional (2D) invertible maps was given in reference [1]. However, there is no corresponding

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theory for multidimensional dynamical systems [1, 8]. In the present work we determine the critical exponents by studying the relation between the power-law scaling for the characteristic times, the eigenvalues of the unstable orbits in the 4LM, and the corresponding eigenvalues of the flow obtained upon application of the centre manifold method to the original 4LM equations. The centre manifold equations (CME) give solutions similar to those of the 4LM for a broad range of parameters and have been derived in reference [7]. A relevant difference is that the 4LM reduces to a 5D invertible map while the CME reduces to a 2D invertible map. Therefore, for the CME we may apply the existing theory [1].

A critical exponent η close $\frac{1}{2}$ is a characteristic of a highly dissipative system [1]. It has been shown that for a certain range of parameters, the 4LM may be described qualitatively by an effective 1D map [6]. This suggests that here the critical exponents $\eta \approx \frac{1}{2}$. However, the 4LM shows properties [6, 7] such as the coexistence of attractors that do not exist in 1D maps. Therefore, in order to determine consistently the critical exponents in the 4LM, the following points need to be considered: (i) the relation between the eigenvalues of the unstable orbits, the critical exponents η and the dynamics at the crises in the 4LM (we consider the three different kind of crises mentioned above in the 4LM); (ii) the extent to which the CME are useful to describe crises in the 4LM. To our knowledge there is no theory that relates the critical exponents of a given model and those of its centre manifold version at crises [8, 9]. We show in this article that both the 4LM and the CME support the conjecture that the existence of critical exponents $\eta \approx \frac{1}{2}$ is a generic property at crises.

On the other hand, saddle-node bifurcations play a relevant role in governing the behaviour of dynamical systems [8, 9], which in our case describe laser systems. Indeed, saddle points can create and destroy attractors [10, 11] or determine the linking properties of chaotic attractors [12]. In this paper, the eigenvalues that we have obtained for the CME are compared with those of the 2LM and no qualitative agreement is found. We underline that by using embedding it is possible to obtain the eigenvalues of the relevant saddle points from experimental data [8].

In the next section we describe the 4LM, the CME and the 2LM. In section three we study the critical exponents in the 4LM and the CME. Further, we find the typical eigenvalues predicted by the 2LM and compare them with those of the previous models. In section four we give the conclusions.

2. Model for the CO_2 laser

The 4LM for the CO_2 laser with modulated losses is based upon the usual field-matter equations for two-level atoms in a resonant cavity [13]. However, the lasing transitions take place between rotational levels which belong to different vibrational bands. These lasing rotational levels are in turn coupled non-radiatively with the other rotational levels of the same vibrational band. The non-lasing rotational levels are assumed to have global populations [5, 14]. When the adiabatic elimination of the polarization is carried out, the 4LM is described by the following equations [5, 6]:

$$\frac{dI}{dt} = -k(t)I + I(N_2 - N_1),$$

$$\frac{dN_2}{dt} = -I(N_2 - N_1) - \gamma_2 N_2 - \gamma_R N_2 + \gamma_R' M_2 + \gamma_2 Q,$$

$$\frac{dN_1}{dt} = I(N_2 - N_1) - \gamma_1 N_1 - \gamma_R N_1 + \gamma'_R M_1,$$

$$\frac{dM_2}{dt} = -\gamma_2 M_2 + \gamma_R N_2 - \gamma'_R M_2 + Z\gamma_2 Q,$$

$$\frac{dM_1}{dt} = -\gamma_1 M_1 + \gamma_R N_1 - \gamma'_R M_1. \quad (1)$$

Here I is the intensity of the electric field, and $N_2(M_2)$ and $N_1(M_1)$ are the upper and lower resonant (non-resonant) rotational levels. In equation (1) the populations N_1, N_2, M_1, M_2, the intensity I and the pump Q have been renormalized with respect to the coupling constant, Z is the effective number of rotational levels in each vibrational band, γ'_R is the rotational relaxation rate for the transitions $M_2 \to N_2$ and $M_1 \to N_1$, γ_R is the rotational relaxation rates for the inverse transitions, and $\gamma_R/\gamma'_R = Z$ [5]. The vibrational relaxation rates in N_1 and N_2 are denoted respectively by γ_1 and γ_2; the same relaxation rates hold for M_1 and M_2, respectively. $k(t) = k_0[1 + m\cos(\omega t + \varphi)]$ is the damping rate for the intensity, m is the modulation amplitude, ω is the modulation frequency and φ is the initial phase. The numerical values of the parameters in equation (1) are given in table 1. We study equation (1) by means of the Poincaré section technique, i.e. we sample the solution at $t = i \times T$, $i = 1, \infty$, where $T = 1/f$ is driving period.

Recently, using techniques of the centre manifold theory, a two-dimensional (2D) flow was derived from equation (1) [7]. This method is a more refined version of the adiabatic elimination of fast variables [7, 15]. Firstly, in this method a suitable change of coordinates in the system is made. Then, an approximation is introduced based on an appropriate smallness parameter, which gives rise to a set of two coupled nonlinear equations [7, 15]. The centre manifold equations (CME) and the 4LM give similar solutions for a broad range of parameters [7]. However, one cannot know a priori that the CME and the 4LM have close critical exponents η, since no theory exists that relates the corresponding exponents in both equations.

The CME are the following [7]:

$$\frac{dY}{dt} = -k(t)Y(1 + bY)(1 + aY) + \Delta Y(1 + bY),$$

$$\frac{d\Delta}{dt} = -\gamma_+ \Delta - 2\Delta Y + \gamma_p Q + k(t)(\tilde{\gamma} + 2aY^2). \quad (2)$$

Here, $Y = I/(Z + 1)$ and $\Delta = (N_2 + M_2 - N_1 - M_1)/(Z + 1)$, $\gamma_+ = (\gamma_1 + \gamma_2)/2$, $\gamma_p = 2\gamma_1\gamma_2/(\gamma_1 + \gamma_2)$, $\tilde{\gamma} = (\gamma_1 - \gamma_2)$, $a = 2Z/\Gamma_R$, and $b = a[k(t)/\Gamma_R - 1]$, where

Table 1. Parameters of the four-level model for the CO_2 laser ($\mu = 10^6\,s^{-1}$).

Parameter	Numerical value	Parameter	Numerical value
λ_R	$7{\cdot}0\mu$	γ'_R	$0{\cdot}7\mu$
γ_1	$0{\cdot}08\mu$	γ_2	$0{\cdot}01\mu$
k_0	$23{\cdot}15\mu$	ω	100 kHz
Q	1·4256	φ	$3\pi/2$
Z	10		

$\Gamma_R = \gamma_R + \gamma'_R$. For the CME, η can be calculated in terms of the eigenvalues [1], since these equations can be reduced to a 2D invertible map. In equation (2) the third dimension appears as a result of the modulated damping rate $k(t)$.

Finally, there is a simpler and popular model for the CO_2 laser known as the two-level model (2LM) [13]. This model is described by the following set of equations:

$$\begin{aligned} \frac{d\tilde{I}}{dt} &= -k(t)\tilde{I} + \tilde{\Delta}\tilde{I}, \\ \frac{d\tilde{\Delta}}{dt} &= -\gamma\tilde{\Delta} - 2\tilde{\Delta}\tilde{I} + \gamma Q, \end{aligned} \tag{3}$$

where $\tilde{I}$ and $\tilde{\Delta}$ stand for the laser intensity and population inversion, respectively. γ is a small parameter and was changed from $\gamma = 10^3$ to 10^5 Hz in order to fit the results of early experiments [13]. Recently, however, it was shown that the 2LM predicts incorrect behaviour for the CO_2 laser in the transient and chaotic regimes of laser operation [5, 6].

3. The critical exponents

Firstly, we describe those crises for which the eigenvalues of the unstable orbits are determined. To this end we have carried out a study of the Poincaré sections in the neighbourhood of the crises reported here. In figure 1 (*a*), we see the bifurcation

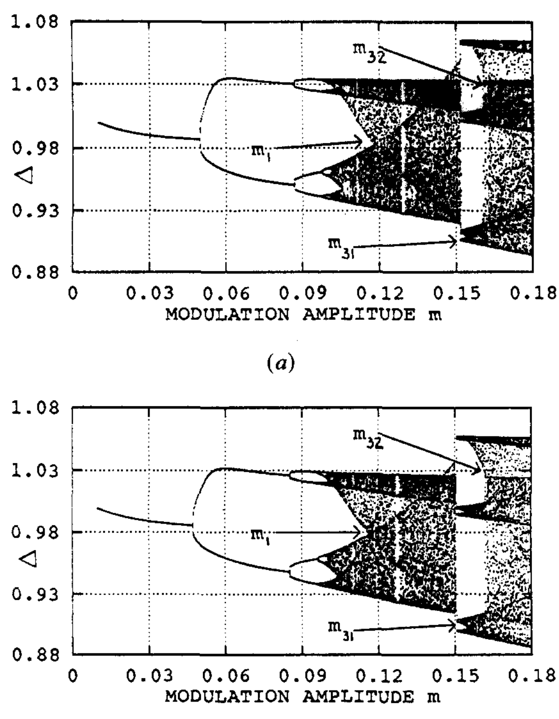

Figure 1. (*a*) Bifurcation diagram for the inversion $\Delta = (N_2 - N_1 + M_2 - M_1)/(Z + 1)$ against the modulation amplitude m in the four-level model (4LM); (*b*) the same as (*a*) but for the centre manifold equations (CME).

diagram for the effective population inversion $\Delta = (N_2 + M_2 - N_1 - M_1)/(Z+1)$ versus the modulation amplitude m in the 4LM. Here, the arrows indicate three different crises in this model. We observe that, starting from low values of m, a period doubling cascade that ends in chaos takes place. The basic period of this cascade is one. After the onset of chaos, merging of chaotic bands takes place and at the end of the merging crises two big attractor pieces in the Poincaré section merge at $m = m_1 \approx 0{\cdot}117\,760$ due to the collision of the attractor with the unstable orbit of period one P_1. The merging of attractors results in homoclinic crises [1]. In this figure we also observe that two successive crises occur at $m = m_{31} \approx 0{\cdot}151\,62$ and at $m = m_{32} \approx 0{\cdot}16028$. The crisis at $m = m_{31}$ is of an attractor destruction type while that at $m = m_{32}$ is of an attractor widening type. Both crises occur due to the collision of the existing attractor with the unstable orbit of period three P_3. These two successive crises due to the same unstable orbit, in our case P_3, are described as the formation of sequential horseshoes in chaotic attractors [9, 16]; both are homoclinic crises [9, 16]. In figure 1 (*b*) we show the corresponding crises in the CME. We see that bifurcation diagrams in figures 1 (*a*) and (*b*) are basically the same.

Near the crises for 1D maps, dissipative 2D maps or 3D dissipative flows, the characteristic time τ is found theoretically to scale with the control parameter m as [1]

$$\tau = a|m - m_c|^{-\eta}. \tag{4}$$

Here $\eta > 0$ is the critical exponent, m_c is the critical parameter and a is a constant. In a 2D invertible map and for a homoclinic tangency, the critical exponents η is given by

$$\eta = \frac{\ln|\beta_2|}{2\ln|\beta_1\beta_2|}, \tag{5}$$

where β_1 is the expanding eigenvalue ($|\beta_1 > 1$) and β_2 is the contracting eigenvalue ($|\beta_2| < 1$) [1].

Figures 2 (*a*) and (*b*) show that the scaling dependence given by equation (3) holds in the 4LM for the attractor merging crisis at $m = m_1$ and for the attractor destruction crisis at $m = m_{31}$, respectively. The corresponding slopes η of the straight lines at $m = m_1$ and $m = m_{31}$ are $\eta_1^{(\mathrm{mod})} = 0{\cdot}502 \pm 1{\cdot}3 \times 10^{-2}$ and $\eta_{31}^{(\mathrm{mod})} = 0{\cdot}529 \pm 1{\cdot}2 \times 10^{-2}$, respectively. If our map was a highly contracting 2D map then by setting $\beta_2 \to 0$ we obtain $\eta = \frac{1}{2}$ from equation (4).

Table 2 shows the eigenvalues for the corresponding unstable orbits at the crises at $m = m_1$, m_{31} and m_{32} in the 4LM. Table 3 contains the same information but for the CME. As regards the 4LM, there are two eigenvalues order $O(1)$, one expansive and another contractive. Then follows an eigenvalue of order $O(10^{-3}$–$10^{-8})$ and finally two highly contractive eigenvalues whose order of magnitude is equal or less than $O(10^{-33}$–$10^{-34})$. The round-off errors in the algorithm limit the estimation of the true value of the last two eigenvalues. We have found that this spectrum of eigenvalues for the unstable orbits is typical for other crises that we have considered in the 4LM. The CME have instead one expanding eigenvalue of order $O(1)$ and one highly contractive eigenvalue, whose order of magnitude is close to λ_3 in the 4LM. Similarly, this spectrum of eigenvalues is also typical for other crises in the CME.

As mentioned above, crises at m_1, m_{31} and m_{32} are homoclinic. If in the 4LM only the most contracting and only the expanding eigenvalues played the leading role at

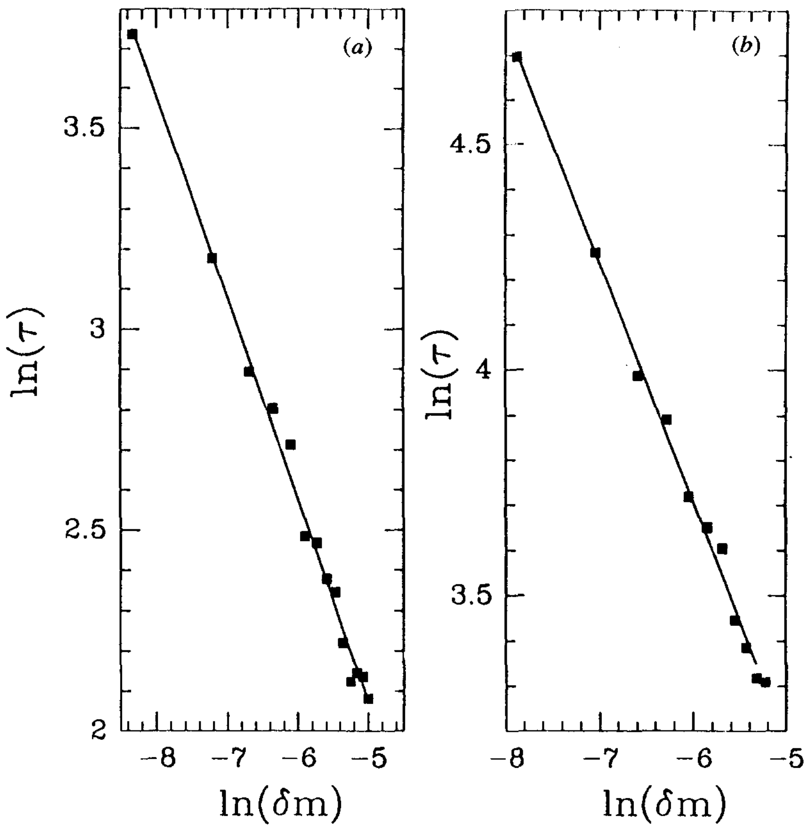

Figure 2. Power-law scaling for the characteristic times τ near crises in the four-level model (4LM). Plots of $\ln \tau$ against $\ln \delta m$ (*a*) $\delta m = m - m_1$ and (*b*) $\delta m = m - m_{31}$. The straight lines are the best fits to the data. The measured critical exponents of the 4LM are $\eta^{(\mathrm{mod})} \approx \frac{1}{2}$ in both cases.

Table 2 Eigenvalues λ_i at the crises $m = m_1$, $m = m_{31}$ and $m = m_{32}$ in the four-level model (equation (1) in the text); the first three eigenvalues are real numbers.

Control parameter	$m = m_1 = 0{\cdot}11770$	$m = m_{31} = 0{\cdot}151162$	$m = m_{32} = 0{\cdot}161$
$\lvert\lambda_1\rvert$	1·88	3·79	4·41
$\lvert\lambda_2\rvert$	0·65	0·28	0·28
$\log_{10}(\lvert\lambda_3\rvert)$	− 3	− 8	− 8
$\log_{10}(\lvert\lambda_{4,5}\rvert)$	≤ − 33	≤ − 34	≤ − 34

Table 3. Eigenvalues λ_i at the crises $m = m_1$, $m = m_{31}$ and $m = m_{32}$ for the centre manifold equations (equation (2) in the text); all the eigenvalues are real numbers.

Control parameter	$m = m_1 = 0{\cdot}115$	$m = m_{31} = 0{\cdot}151$	$m = m_1 = 0{\cdot}161$
$\lvert\lambda_1\rvert$	1·88	1·41	1·71
$\log_{10}(\lvert\lambda_2\rvert)$	− 3	− 10	− 10

crises, then it would be enough to replace $\beta_1 \to \lambda_1$ and $\beta_2 \to \lambda_{4,5}$ to obtain $\eta \approx \frac{1}{2}$ in equation (4). Namely, with the ansatz $\beta_1 \to \lambda_1$, $\beta_2 \to \lambda_{4,5}$, we obtain $0{\cdot}5 < \eta_1^{(\text{theor})} < 0{\cdot}504$ at $m = m_1$, $0{\cdot}5 < \eta_{31}^{(\text{theor})} < 0{\cdot}509$ at $m = m_{31}$ and $0{\cdot}5 < \eta_{32}^{(\text{theor})} < 0{\cdot}510$ at m_{32}. In this manner a good agreement between $\eta_{1,31}^{(\text{mod})}$ and $\eta_{1,31}^{(\text{theor})}$ is found. On the other hand, if only expanding eigenvalue and the contracting eigenvalue λ_3 were the only relevant eigenvalues at the crises, then $\beta_1 \to \lambda_1$, $\beta_2 \to \lambda_3$, $\eta_1 = 0{\cdot}550$, $\eta_{31} = 0{\cdot}508$, and the agreement with $\eta^{(\text{mod})}$ is not as good as in the previous ansatz.

This ansatz suggests that near the crises the most contracting eigenvalues play a relevant role in highly dissipative systems. In the 4LM, other crises due to the unstable orbits of period four, five and six within the interval $0 < m < 0{\cdot}25$ have their spectrum of eigenvalues similar to those of table 2. Therefore, we believe that the dynamics at other homoclinic crises must be the same as at $m = m_1$ and at $m = m_{31}$, and here the critical exponents are also $\eta \approx \frac{1}{2}$.

On the other hand, the CME predict critical exponents $\eta \approx \frac{1}{2}$. This can be proved by inserting the eigenvalues given in table 3 into equation (4). We obtain $\eta_1 = 0{\cdot}550$, $\eta_{31} = 0{\cdot}507$ and $\eta_{32} = 0{\cdot}512$ for m_1, m_{32} and m_{32}, respectively. These values η are close to $\eta^{(\text{mod})}$. The ubiquity of a highly contracting eigenvalue at all considered crises in the CME and the similarity between the CME and the 4LM as shown in figures 1 (*a*) and (*b*) suggest that at homoclinic and heteroclinic crises $\eta \approx \frac{1}{2}$ in the multidimensional 4LM. This similarity between the 4LM and the CME dynamics and the existence of critical exponents in the CME, may be considered as indirect proof that the power-law scaling holds in the multidimensional 4LM.

Let us compare the above-mentioned results with those of the 2LM with modulated losses. This model was extensively studied theoretically in [10, 11]. It was shown that for $\gamma \approx 10^3$ Hz as claimed in early experiments [13], the behaviour of the 2LM is that of a conservative system with a small dissipative perturbation [10, 11]. Therefore, a small external periodic modulation may stabilize the damped oscillations of the system [10, 11]. On the other hand, in a conservative 2D map the eigenvalues of saddles are $|\lambda_1| > 1$ and $|\lambda_2| < 1$ with $|\lambda_2 \lambda_2| = 1$. In order to fit the results of the experiments γ was chosen to vary from 10^4 to 10^5 Hz [5, 6]. However, recent studies [5, 6] proved that even these new values cannot reproduce important features of the experiment.

As stated above the behaviour of a dynamical system depends critically on its periodic orbits. Therefore, it is interesting to know the typical eigenvalue spectrum predicted by the 2LM. This spectrum corresponds to the parameter region of m around which the first attractor destruction crisis occurs in the 2LM. For $\gamma = 1{\cdot}95 \times 10^4$ Hz we obtain for the period one and period two unstable orbits the following spectrum. For saddles between $0{\cdot}016 < m < 0{\cdot}02$, $|\lambda_1| \sim O(1) > 1$ and $|\lambda_2| \sim O(1) < 1$, e.g. for $m = 0{\cdot}019$, $|\lambda_1| \approx 1{\cdot}5$ and $|\lambda_2| \approx 0{\cdot}4$. On the other hand, for $\gamma = 1{\cdot}95 \times 10^5$ Hz, for the period one orbit the eigenvalues of the saddle at $m = 0{\cdot}155$ are $|\lambda_1| \approx 1{\cdot}3$ and $|\lambda_2| \approx 0{\cdot}13$, while for the period two orbit the eigenvalues of the saddle at $m = 0{\cdot}154$ are $|\lambda_1| \approx 1{\cdot}3$ and $|\lambda_2| \approx 5 \times 10^{-3}$. These eigenvalues are typical for $0{\cdot}15 < m < 0{\cdot}16$.

It is known that in the 2LM, the attractor originating from the period one orbit undergoes a first attractor destruction crisis caused by the unstable orbit of period two [6, 10]. Instead in the 4LM as well as in the CME, the bifurcation diagram is close to that of the logistic map [17]. The latter is certainly at odds with the 2LM predictions. Moreover, for $\gamma = 1{\cdot}95 \times 10^4$ Hz, the contracting eigenvalues for the

2LM are too big when compared to those of the CME. Therefore, the predicted η is substantially different from $\frac{1}{2}$. On the other hand for $\gamma = 1{\cdot}95 \times 10^5$ Hz, the contracting eigenvalues of the 2LM for period one are two orders of magnitude larger than those for the CME, for the same range of values of m. On the basis of the above study, we believe that an experiment can be carried out in the following way. Since, sweeping the modulation depth m and measuring the attactor lifetimes is followed by strong noise in the experiment [18], the straightforward method of finding the eigenvalues seems more appropriate to test our results. To this end we proceed by using embedding techniques [8]. As a result we form the delay coordinate vector $\bar{\varphi}(t) = \{\varphi(t), \varphi(t - T_d), \dots, \varphi[t - (Q - 1)T_d]\}$, where T_d is the delay time, Q is the embedding dimension [8], and $\varphi(t)$ is the laser intensity. Typically for low dimensional attractors $T_d \simeq 0{\cdot}2\, T$, where T is the basic period on the attractor. Next, in order to obtain a discrete time series from $\bar{\varphi}(t)$ we can use a Poincaré surface of section but in $\bar{\varphi}$-space. Let Z_n be the points in the $\bar{\varphi}$-space surface of section for a given value of the control m. In order to find the eigenstates of an unstable periodic orbit, say of period one, we must find a matrix $\mathbf{A}$ such that

$$Z_{n+1} = \mathbf{A}Z_n + C,$$

for all possible pairs Z_n and Z_{n+1} lying close to some Z^*, which is the period one orbit. $\mathbf{A}$ is the least-squares fit matrix and is an approximation to Jacobian matrix $\mathbf{A}$ of the exact map from where we obtain the eigenvalues of the periodic orbit [19].

The calculation of periodic orbits and eigenvalues was performed using standard methods for the appropriate nth iterate of the Poincaré map [20, 21]. To determine the characteristic times for these crises at $m = m_1$ and at $m = m_{31}$, a binary partition was chosen as suggested by figure 1 (a). We used 300 initial conditions localized at $t = 0$ in the core ($m = m_1$) or old ($m = m_{31}$) attractors. We chose as the critical bifurcation parameter m_c, that with the least error for $\eta_1^{(\mathrm{mod})}$ and $\eta_{31}^{(\mathrm{mod})}$ when fitting data to a straight line, as shown in figure 2.

4. Conclusions

Our results can be divided into two parts. First, we have shown that a multidimensional dynamical system known as the four-level model (4LM) which describes the CO_2 laser with modulated losses predicts a power-law scaling for the characteristic times in three different crises. These are attractor merging, attractor destruction and attractor widening crises. The mean lifetimes of chaotic attractors at crises are related to the critical exponents. Our study suggests that the expanding and the most contracting eigenvalues play a relevant role in the dynamics at the crises in the 4LM. Our work was based on a direct determination of the critical exponents and the corresponding eigenvalues at the crises in the 4LM. Morever, the centre manifold equations (CME) derived from the 4LM predict that one of its two eigenvalues is highly contracting. As a result the critical exponents of the CME are in agreement with those of the 4LM. The spectra of the eigenvalues in the 4LM are qualitatively the same at all crises that we have considered. The same takes place in the CME, This property of the spectra of the eigenvalues and the fact that the dynamics of the 4LM and the CME is similar suggest that the existence of critical exponents $\eta \approx \frac{1}{2}$ is a generic property for all homoclinic and heteroclinic crises in the 4LM, at least within the range of control parameters that we have studied ($0 < m < 0{\cdot}25$, where m is the modulation amplitude).

Secondly, the simpler and popular model for the CO_2 laser known at the two-level model has eigenvalues for its unstable periodic orbits that differ qualitatively from those predicted by the CME. The comparison is made in the parameter region of *m* where the first attractor destruction crisis in the two-level model takes place. This region is interesting since, it allows us to easily find relevant differences in the above-mentioned models in terms of their bifurcation diagrams and with those of the experiment [6]. In the experiment, embedding techniques are necessary to calculate the eigenvalues of the unstable periodic orbits. We conclude that a practical way to find the attractor lifetimes is to obtain these eigenvalues in the experiment.

We believe that the present work may be useful in the study of crisis in related multidimensional laser systems which show high dissipation. Finally, our numerical calculations indicate that it would be useful to have a theoretical estimate of the critical exponents (attractor lifetimes) of a multidimensional system in terms of those of its centre manifold version.

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References

[1] GREBOGI, C., OTT, E., and YORKE, J. A., 1986, *Phys. Rev. Lett.*, **57,** 1284; GREBOGI, C., OTT, E., ROMEIRAS, F., and YORKE, J. A., 1987, *Phys. Rev.* A, **36,** 5365.
[2] CARROL, T. L., PECORA, L. M., and RACHFORD, F. J., 1987, *Phys. Rev. Lett.*, **59,** 2891; SOMMERER, J. C., DITTO, W. L., GREBOGI, C., OTT, E., and SPANO, M., 1991, *Phys. Lett.* A, **153,** 105 (and reference therein).
[3] FLEPP, L., HOLZNER, R., BRUN, E., FINARDI, M., and BADDI, R., 1991, *Phys. Rev. Lett.*, **67,** 2244; FINARDI, M., FLEPP, L., PARISI, J., HOLZNER, R., BADII, R., and BRUN, E., 1992, *Phys. Rev. Lett.*, **68,** 2989.
[4] SOMMERER, J. C., OTT, E., and GREBOGI, C., 1991, *Phys. Rev.* A, **43,** 1754.
[5] MEUCCI, R., CIOFINI, M., and WANG, P., 1992, *Optics Commun.*, **91,** 444; ZEHNLE, V., DANGOISSE, D., and GLORIEUX, P., 1992, *Optics Commun.*, **90,** 99.
[6] PANDO, C. L., MEUCCI, R., CIOFINI, M., and ARECCHI, F. T., 1993, *Chaos*, **3,** 279.
[7] CIOFINI, M., POLITI, A., and MEUCCI, R., 1993, *Phys. Rev.* A, **48,** 605.
[8] OTT, E., 19932, *Chaos in Dynamical Systems* (Cambridge: CUP).
[9] WIGGINS, S., 1990, *Introduction to Nonlinear Dynamical Systems and Chaos* (New York: Springer.Verlag); CRAWFORD, J. D., 1991, *Rev. mod. Phys.*, **63,** 991.
[10] SCHWARTZ, I. B., and ERNEUX, T., 1995, *Siam. J. appl. Math.* (to be published).
[11] ERNEUX, T., BAER, S. M., and MANDEL, P., 1987, *Phys. Rev.* A, **35,** 1165.
[12] MINDLIN, G. B., HOU, X.-J., SOLARI, H. G., GILMORE, R., and TUFILLARO, N. B., 1990, *Phys. Rev. Lett.*, **64,** 2350.
[13] ARECCHI, F. T., 1987, *Instabilities and Chaos in Quantum Optics*, edited by F. T. Arecchi and R. G. Harrison (Berlin: Springer-Verlag); ABRAHAM, N. B., MANDEL, P., and NARDUCCI, L., 1988, *Prog. Optics*, **25,** 1.
[14] DUPRE, J., MEYER, F., and MEYER, C., 1975, *Rev. Phys. Appl.*, **10,** 285; ARIMONDO, E., CASAGRANDE, F., LUGIATO, L. A., and GLORIEUX, P., 1983, *Appl. Phys.* B, **30,** 57.
[15] OPPO, G. L., TREDICCE, J. R., and NARDUCCI, L. M., 1989, *Optics Commun.*, **69,** 393; OPPO, G. L., and POLITI, A., 1989, *Phys. Rev.* A, **40,** 1422.
[16] SCHWARTZ, I. B., 1988, *Phys. Rev. Lett.*, **60,** 1359.
[17] PANDO, C. L., LUNA-ACOSTA, G. A., MEUCCI, R., and CIOFINI, M., 1995, *Phys. Lett.* A, **199,** 191.
[18] SHINBROT, T., GREBOGI, C., OTT, E., and YORKE, J. A., 1993, *Nature*, **363,** 411.
[19] SPARROW, C. T., 1982, *The Lorenz Equations* (Berlin: Springer-Verlag).
[20] PRESS, W. H., FLANNERY B. P., TEUKOLSKY, S. A., and VETTERING, W. T., 1989, *Numerical Recipes* (Cambridge: CUP).