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Infinite set of relevant operators for an exact solution of the time-dependent Jaynes-Cummings Hamiltonian

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The dynamics and thermodynamics of a quantum time-dependent field coupled to a two-level system, well known as the Jaynes-Cummings Hamiltonian, is studied, using the maximum entropy principle. In the framework of this approach we found three different infinite sets of relevant operators that describe the dynamics of the system for any temporal dependence. These sets of relevant operators are connected by isomorphisms, which allow us to consider the case of mixed initial conditions. A consistent set of initial conditions is established using the maximum entropy principle density operator, obtaining restrictions to the physically feasible initial conditions of the system. The behavior of the population inversion is shown for different time dependencies of the Hamiltonian and initial conditions. For the time-independent case, an explicit solution for the population inversion in terms of the relevant operators of one of the sets is given. It is also shown how the well known formulas for the population inversion are recovered for the special cases where the initial conditions correspond to a pure, coherent, and thermal field.

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I. INTRODUCTION

The problem of a two-level system coupled to a single mode of a radiation field [1] is the simplest nontrivial model which describes the matter-radiation interaction. In the rotating wave approximation this model becomes exactly solvable and describes a large amount of phenomena in fields such as quantum optics, NMR, and quantum electronics [2–9]. Since the paper of Jaynes and Cummings [1], many authors have studied the collapses and revivals of the population inversion when the field is initially in a coherent state. In particular, Eberly *et al.* [10] found accurate expressions to describe the intermediate and long time behavior of the population inversion. Experimental evidence of this phenomena has also been observed [11,12]. Recently, the thermodynamics of this model has been studied by Liu and Tombesi [13]; departing from the grand partition function, they found the temperature dependence for some thermodynamical magnitudes. One of the fundamental features that appears in the quantum two-level system is the complex structure of the correlation between the two-level system and the field due to the quantum character of the latter [14–16]. The maximum entropy principle (MEP) approach [17–19] has already been used to solve the two-level system coupled to a classical field [20,21]. In that work the complete dynamics was described in terms of the mean value of four relevant operators. A consistent set of initial conditions (CSIC) [20] was established using the MEP density operator, obtaining restrictions to the physically feasible initial conditions (IC) of the system.

In this paper we study a generalization of the quantum two-level system to the time-dependent case [14,22–27]. Since we are interested in obtaining a description to be used in different fields of physics, we shall describe the system in terms of three different sets of relevant operators, which are straightforwardly obtained by use of the MEP. These sets of operators are connected via linear transformations which allows us to change from one set to another. For the Jaynes-Cummings Hamiltonian (JCH) *the operator sets are infinite* and in consequence the dynamics is described by an *infinite set of ordinary differential equations* for the mean values of the relevant operators, making evident the quantum character of the field. As we have shown before, the IC for the dynamical set of equations cannot be arbitrarily chosen [17,18]. The CSIC is properly obtained using the MEP density matrix. The purpose is achieved introducing the Hamiltonian as a relevant operator for constructing our density matrix. This fact, leads to a quantum thermodynamical description of the problem [28]. The time-independent case is studied in detail in order to reduce our general results to previous ones. A general solution for the population number of one of the levels is obtained and the results for a pure, coherent, and thermal state are recovered [29–32]. We compare an approximate time evolution of the mean value of the level population with the one obtained by the series solution, when the field is initially

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in a coherent state, in order to evaluate the effects of the approximation on the evolution equations. It is observed that the approximate solution is improved, and tends to the series solution, when the number of correlations included in the solution is increased. Finally, for the time-dependent case, we recover the results obtained by Prants and Yacoupova [25] and Joshi and Lawande [26] for their functions of time when the field is initially in a coherent state. The same time dependencies are also studied and extended to the thermal case and the results are discussed.

The paper is organized as follow: In Sec. II we give some concepts of the MEP approach; Sec. III is devoted to obtaining the three sets of relevant operators, giving the transformations between sets and the dynamical equation for one of these sets. In Sec. IV the thermodynamical treatment is developed and invariants of motion are presented. Section V gives the exact solution of the population inversion for a general IC. We also show how the formulas for the population inversion are recovered when a pure, coherent, and thermal state is considered. In Sec. VI we give the numerical results obtained for the population inversion and the second-order coherence function for a linear and an exponential time dependence when the field is initially in a coherent or thermal state. Finally, in Sec. VII the conclusions are drawn.

II. MEP CONCEPTS

We summarize in this section the fundamental concepts of the MEP [17,18]. Given the expectation values $\langle \hat{O}_j \rangle$ of the operators $\hat{O}_j$, the statistical operator $\hat{\rho}(t)$ is defined by

$$\hat{\rho}(t) = \exp\left(-\lambda_0 \hat{I} - \sum_{j=1}^{M} \lambda_j \hat{O}_j\right) , \tag{2.1}$$

where M is a natural number or infinity, and the $M+1$ Lagrange multipliers λ_j are determined to fulfill the set of constraints

$$\langle \hat{O}_j \rangle = \mathrm{Tr}\,[\,\hat{\rho}(t)\,\hat{O}_j\,] , \qquad j = 0, 1, \ldots, M \tag{2.2}$$

($\hat{O}_0 = \hat{I}$ is the identity operator). The entropy, defined in units of the Boltzmann constant, is given by

$$S(\hat{\rho}) = -\mathrm{Tr}\,[\,\hat{\rho}\,\ln \hat{\rho}\,] = \lambda_0 \hat{I} + \sum_{j=1}^{M} \lambda_j \langle \hat{O}_j \rangle, \tag{2.3}$$

and the time evolution of the statistical operator is given by

$$i\hbar \frac{d\hat{\rho}}{dt} = [\,\hat{H}(t), \hat{\rho}(t)\,] . \tag{2.4}$$

One should find those relevant operators (RO) entering in Eq. (2.1) so as to guarantee not only that S is maximum, but also a constant of motion. Introducing the natural logarithm of Eq. (2.1) into Eq. (2.4) it can be easily verified that the RO are those that close a semi-Lie algebra under commutation with the Hamiltonian $\hat{H}$, i.e.,

$$[\,\hat{H}(t), \hat{O}_i\,] = i\hbar \sum_{j=0}^{L} g_{ji}(t) \hat{O}_j . \tag{2.5}$$

Equation (2.5) defines an $L \times L$ matrix G and constitutes the central requirement to be fulfilled by the operators entering in the density matrix. The Liouville equation (2.4) can be replaced by a set of coupled equations for either the mean values of the relevant operators or the Lagrange multipliers as follows [17,18]:

$$\frac{d\langle \hat{O}_j \rangle_t}{dt} = -\sum_{i=0}^{L} g_{ij} \langle \hat{O}_i \rangle , \qquad j = 0, 1, \ldots, L , \tag{2.6}$$

$$\frac{d\lambda_j}{dt} = \sum_{i=0}^{L} \lambda_i g_{ji} , \qquad j = 0, 1, \ldots, L . \tag{2.7}$$

In the MEP formalism, the mean value of the operators and the Lagrange multipliers belong to dual spaces which are connected by [17,18],

$$\langle \hat{O}_j \rangle = -\frac{\partial \lambda_0}{\partial \lambda_j} . \tag{2.8}$$

III. TIME-DEPENDENT JCH AND PHYSICALLY RELEVANT OPERATORS

The *generalized time-dependent* JCH in the rotating wave approximation takes the form

$$\begin{aligned} \hat{H} = {} & E_1 \hat{b}_1^\dagger \hat{b}_1 + E_2 \hat{b}_2^\dagger \hat{b}_2 + \omega \hat{a}^\dagger \hat{a} \\ & + h(t)\left(\gamma \hat{a} \hat{b}_1 \hat{b}_2^\dagger + \gamma^* \hat{b}_2 \hat{b}_1^\dagger \hat{a}^\dagger\right), \end{aligned} \tag{3.1}$$

($\hbar = 1$), where γ is the coupling constant between the system and the external field, E_j and ω are the energies of the levels and the field, respectively, $\hat{a}^\dagger$ and $\hat{a}$ are boson operators, $\hat{b}_j^\dagger$ and $\hat{b}_j$ are fermion operators and $h(t)$ is an arbitrary (adimensional) function of time. As was said in Sec. II the MEP approach is based on a description of the problem in terms of *relevant operators* which should close a semi-Lie algebra under commutation with the Hamiltonian. In the problem considered in this paper it turns out that the relevant operators can be presented in *three different but equivalent sets*, each of them having different physical interpretations and connected via isomorphisms which allow us to go from one set to another.

A. Sets of relevant operators

The advantage of having multiple representations (in our case three different sets of relevant operators connected by isomorphisms) is clearly seen when partial information in any set is known (e.g., some initial mean values are unknown), and it is possible to complete the missing information using the complementary data in any other set (i.e., mixed IC [18]). Thus, first considering the level populations, it is found that the infinite sets of RO can be written in terms of only six elementary operators which will be called physically relevant operators. They are

$$\hat{N}_1 \equiv \hat{b}_1^\dagger \hat{b}_1 \ , \tag{3.2a}$$

$$\hat{N}_2 \equiv \hat{b}_2^\dagger \hat{b}_2 \ , \tag{3.2b}$$

$$\hat{\Delta} \equiv \hat{a}^\dagger \hat{a} \ , \tag{3.2c}$$

$$\hat{I} \equiv \gamma \hat{a} \hat{b}_1 \hat{b}_2^\dagger + \gamma^* \hat{b}_2 \hat{b}_1^\dagger \hat{a}^\dagger \ , \tag{3.2d}$$

$$\hat{F} \equiv i(\gamma \hat{a} \hat{b}_1 \hat{b}_2^\dagger - \gamma^* \hat{b}_2 \hat{b}_1^\dagger \hat{a}^\dagger) \ , \tag{3.2e}$$

$$\hat{N}_{2,1} \equiv \hat{b}_2^\dagger \hat{b}_2 \hat{b}_1^\dagger \hat{b}_1 \ , \tag{3.2f}$$

where $\hat{N}_l$, l =1, 2, and $\hat{\Delta}$ can be thought of as the population number of the levels and the external field, respectively. $\hat{I}$ can be considered as the interaction energy between the levels and the external field, $\hat{F}$ as the particle current between levels and, finally, $\hat{N}_{2,1}$ as the double occupation number. It is interesting to mention that the operators [(3.2a), (3.2b), (3.2d)–(3.2f)] can be considered as the quantum counterpart of the operators obtained for the semiclassical two-level system studied in Ref. [20].

The first set of RO, SI, which closes a semi-Lie algebra with the Hamiltonian [see Eq. (2.5)] reads,

$$\hat{N}_1^n \equiv (\hat{a}^\dagger)^n \ \hat{N}_1 \ (\hat{a})^n, \tag{3.3a}$$

$$\hat{N}_2^n \equiv (\hat{a}^\dagger)^n \ \hat{N}_2 \ (\hat{a})^n, \tag{3.3b}$$

$$\hat{\Delta}^n \equiv (\hat{a}^\dagger)^n \ \hat{\Delta} \ (\hat{a})^n, \tag{3.3c}$$

$$\hat{I}^n \equiv (\hat{a}^\dagger)^n \ \hat{I} \ (\hat{a})^n \ , \tag{3.3d}$$

$$\hat{F}^n \equiv (\hat{a}^\dagger)^n \ \hat{F} \ (\hat{a})^n \ , \tag{3.3e}$$

$$\hat{N}_{2,1}^n \equiv (\hat{a}^\dagger)^n \ \hat{N}_{2,1} \ (\hat{a})^n \ , \tag{3.3f}$$

$n = 0, 1, \ldots$. The RO of this set have the main property of being in *normal order* [33]. For $n = 0$ Eqs. (3.3) reduce to the fundamental set [Eqs. (3.2)]. This set is suitable for numerical simulation because it provides the simplest form of the system of differential equations for the evolution of their mean values.

The second infinite set of RO, SII, which also fulfills the closure relation Eq. (2.5) has the form

$$\hat{\mathbb{N}}_1^n \equiv \hat{N}_1 \ (\hat{a}^\dagger)^n (\hat{a})^n, \tag{3.4a}$$

$$\hat{\mathbb{N}}_2^n \equiv \hat{N}_2 \ (\hat{a}^\dagger)^n (\hat{a})^n, \tag{3.4b}$$

$$\hat{\mathbb{D}}^n \equiv \frac{1}{2}\left[\hat{\Delta} \ (\hat{a}^\dagger)^n (\hat{a})^n + (\hat{a}^\dagger)^n (\hat{a})^n \hat{\Delta}\right] \ , \tag{3.4c}$$

$$\hat{\mathbb{I}}^n \equiv \frac{1}{2}\left[\hat{I} \ (\hat{a}^\dagger)^n (\hat{a})^n + (\hat{a}^\dagger)^n (\hat{a})^n \hat{I}\right] \ , \tag{3.4d}$$

$$\hat{\mathbb{F}}^n \equiv \frac{1}{2}\left[\hat{F} \ (\hat{a}^\dagger)^n (\hat{a})^n + (\hat{a}^\dagger)^n (\hat{a})^n \hat{F}\right] \ , \tag{3.4e}$$

$$\hat{\mathbb{N}}_{2,1}^n \equiv \hat{N}_{2,1} \ (\hat{a}^\dagger)^n (\hat{a})^n \ , \tag{3.4f}$$

$n = 0, 1, \ldots$. This set can be interpreted as *the correlation functions between the fundamental operators and* $(\hat{a}^\dagger)^n (\hat{a})^n$, which is proportional to the nth-order coherence function of the field (see for example, Ref. [30]). The last set of RO, SIII, describes *the correlations between the fundamental operators and the energy of the field*

$$\hat{\mathfrak{N}}_1^n \equiv \hat{N}_1 \ (\hat{a}^\dagger \hat{a})^n, \tag{3.5a}$$

$$\hat{\mathfrak{N}}_2^n \equiv \hat{N}_2 \ (\hat{a}^\dagger \hat{a})^n, \tag{3.5b}$$

$$\hat{\mathfrak{D}}^n \equiv \hat{\Delta} \ (\hat{a}^\dagger \hat{a})^n \ , \tag{3.5c}$$

$$\hat{\mathfrak{I}}^n \equiv \frac{1}{2}\left[\hat{I} \ (\hat{a}^\dagger \hat{a})^n + (\hat{a}^\dagger \hat{a})^n \ \hat{I}\right] \ , \tag{3.5d}$$

$$\hat{\mathfrak{F}}^n \equiv \frac{1}{2}\left[\hat{F} \ (\hat{a}^\dagger \hat{a})^n + (\hat{a}^\dagger \hat{a})^n \ \hat{F}\right] \ , \tag{3.5e}$$

$$\hat{\mathfrak{N}}_{2,1}^n \equiv \hat{N}_{2,1} \ (\hat{a}^\dagger \hat{a})^n \ , \tag{3.5f}$$

$n = 0, 1, \ldots$.

As mentioned previously the three sets are connected by linear transformations which allow us to transform operators of any set into that of another. These relations between sets give the possibility to consider CSIC where the mean values of the relevant operators belongs to different sets (i.e., mixed initial values). The transformation between SI and SII reads

$$\hat{N}_1^n = \hat{\mathbb{N}}_1^n, \tag{3.6a}$$

$$\hat{N}_2^n = \hat{\mathbb{N}}_2^n, \tag{3.6b}$$

$$\hat{\Delta}^n = \sum_{r=0}^{n} \frac{(-1)^{n-r} n!}{r!} \hat{\mathbb{D}}^r, \tag{3.6c}$$

$$\hat{F}^n = \sum_{r=0}^{n} \frac{(-1)^{n-r} n!}{2^{n-r} r!} \hat{\mathbb{F}}^r \ , \tag{3.6d}$$

$$\hat{I}^n = \sum_{r=0}^{n} \frac{(-1)^{n-r} n!}{2^{n-r} r!} \hat{\mathbb{I}}^r \ , \tag{3.6e}$$

$$\hat{N}_{1,2}^n = \hat{\mathbb{N}}_{1,2}^n. \tag{3.6f}$$

Notice that Eqs. (3.6a), (3.6b), and (3.6f) evidence the fact that *the level populations and the double occupation number commute with the field operators.* On the other hand, since $\hat{F}^n$, $\hat{I}^n$, and $\hat{\Delta}^n$ contain creation and annihilation operators of the field in their definitions, the transformations involve operators from order zero up to n. The transformations between SI and SIII are as follows:

$$\hat{N}_1^n = \sum_{r=0}^{n} (p_{n-r}) \, \hat{\mathfrak{N}}_1^r \ , \tag{3.7a}$$

$$\hat{N}_2^n = \sum_{r=0}^{n} (p_{n-r}) \, \hat{\mathfrak{N}}_2^r \ , \tag{3.7b}$$

$$\hat{\Delta}^n = \sum_{r=0}^{n} \left(\sum_{j=r}^{n} \frac{(-1)^{n-j} \ n! \ p_{j-r}}{j!} \right) \hat{\mathfrak{D}}^r \ , \tag{3.7c}$$

$$\hat{F}^n = \sum_{r=0}^{n} \left(\sum_{j=r}^{n} \frac{(-1)^{n-j} \ n! \ p_{j-r}}{2^{n-j} j!} \right) \hat{\mathfrak{F}}^r \ , \tag{3.7d}$$

$$\hat{I}^n = \sum_{r=0}^{n} \left(\sum_{j=r}^{n} \frac{(-1)^{n-j} \ n! \ p_{j-r}}{2^{n-j} j!} \right) \hat{\mathfrak{I}}^r \ , \tag{3.7e}$$

$$\hat{N}_n^{1,2} = \sum_{r=0}^{n} (p_{n-r})\, \hat{\mathfrak{N}}_{1,2}^{r} \;, \tag{3.7f}$$

where

$$p_{j-r} = \sum_{\substack{0\le i_1<\cdots<i_{j-r}\le j-1 \\ i_1\neq\cdots\neq i_{j-r}}} (-1)^{j-r} i_1 \cdots i_{j-r} \;,$$

$$j \ge r, \quad j,\, r \in \mathrm{N}_0, \quad \text{and} \quad p_0 = 1. \tag{3.8}$$

Since only properties of the field operators were involved in the calculations, the transformations are independent of *the characteristics of the two-level system and the temporal dependence* $h(t)$. From now on, in order to study the dynamical and thermodynamical features of this system, we will deal with the set SI defined by Eqs. (3.3), as it is the easiest to handle in numerical simulations.

In quantum optics, one usually deals with the so-called population inversion given by

$$\langle\hat{\sigma}\rangle_t = \langle\hat{N}_2^0\rangle_t - \langle\hat{N}_1^0\rangle_t \;. \tag{3.9}$$

Another measure of the nonclassical characteristic of the problem is the second-order coherence function,

$$g^2(t) = \frac{\langle(\hat{a}^\dagger)^2\hat{a}^2\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle^2} - 1, \tag{3.10}$$

note that $g^2(t) < 0$ implies sub-Poissonian distribution or antibunching. Thus, if one has only interest in the evolution of the population difference between the two-level the Jaynes-Cummings model in terms of the Pauli matrices can be used. In this case a different set of RO that satisfies Eq. (2.5) is obtained

$$\hat{\sigma}^n = (\hat{a}^\dagger)^n\, \hat{\sigma}\, (\hat{a})^n, \tag{3.11}$$

$$\hat{G}^n = (\hat{a}^\dagger)^n (\hat{a})^n \equiv \hat{\Delta}^{n-1}, \tag{3.12}$$

$$\hat{I}^n = (\hat{a}^\dagger)^n\, (\gamma\hat{a}\hat{\sigma}_+ + \gamma^*\hat{\sigma}_-\hat{a}^\dagger)\, (\hat{a})^n \;, \tag{3.13}$$

$$\hat{F}^n = (\hat{a}^\dagger)^n\, i(\gamma\hat{a}\hat{\sigma}_+ - \gamma^*\hat{\sigma}_-\hat{a}^\dagger)(\hat{a})^n \;, \tag{3.14}$$

where $n = 0, 1, \ldots$. It is important to mention that this set is not equivalent to SI, SII, or SIII because it has less information since we are looking at the difference of population between levels rather than at the population of each level.

Finally, we want to point out that the words *physically relevant operators* has a deep meaning in our context, since different sets emerge from the particular operator structure of the Hamiltonian. For instance, different operator sets are obtained applying Eq. (2.5) if the Hamiltonian is considered with or without the rotating wave approximation.

B. Dynamical equations and invariants of motion

In the MEP formalism the dynamics of the Hamiltonian is described in terms of a set of ordinary differential equations for the mean value of the relevant operators [see Eqs. (2.6)]. Usually this set of ordinary differential equations is of finite dimension and, therefore, is straightforwardly solvable. In the case considered here, *due to the quantum character of the field*, the system becomes *infinite* and with time-dependent coefficients [due to $h(t)$]. However, the Ehrenfest theorem [Eq. (2.6)] is still valid and we obtain the dynamical equations for the *generalized time-dependent* JCH

$$\frac{d\langle\hat{N}_1^n\rangle}{dt} = h(t)\langle\hat{F}^n\rangle + nh(t)\langle\hat{F}^{n-1}\rangle, \tag{3.15a}$$

$$\frac{d\langle\hat{N}_2^n\rangle}{dt} = -h(t)\langle\hat{F}^n\rangle, \tag{3.15b}$$

$$\begin{aligned}\frac{d\langle\hat{F}^n\rangle}{dt} = {} & -\theta\langle\hat{I}^n\rangle + 2|\gamma|^2 h(t)[(n+1)\langle\hat{N}_2^n\rangle \\ & -\langle\hat{N}_1^{n+1}\rangle + \langle\hat{N}_2^{n+1}\rangle \\ & -(n+1)\langle\hat{N}_{2,1}^n\rangle],\end{aligned} \tag{3.15c}$$

$$\frac{d\langle\hat{I}^n\rangle}{dt} = \theta\langle\hat{F}^n\rangle, \tag{3.15d}$$

$$\frac{d\langle\hat{\Delta}^n\rangle}{dt} = (n+1)h(t)\langle\hat{F}^n\rangle, \tag{3.15e}$$

$$\frac{d\langle\hat{N}_{2,1}^n\rangle}{dt} = 0, \tag{3.15f}$$

$n = 0, 1, \ldots$, where $\theta = E_2 - E_1 - \omega$.

As will be seen later, this system can be explicitly solved for the time-independent case recovering well-known previous results. For the time-dependent case, it is also possible to obtain some analytical solutions (see for instance Refs. [14,22–27]), but these analytical expressions should also be approximated to obtain numerical results. In Sec. VII we show that it is possible to obtain numerically exact solutions of the same quality by considering correlations up to a given order. Notice that the dynamical equations [Eqs. (3.15)] can be thought of as a kind of *generalized Bloch equations for the quantum field case.* One interesting point concerning the dynamics of this Hamiltonian is the evaluation of its *invariants of motion.* These can be proved to be

$$\left\{\langle\hat{N}_1^n\rangle_t + \langle\hat{N}_2^n\rangle_t - \langle\hat{\Delta}^{n-1}\rangle_t\right\}_{n=0}^{\infty}, \tag{3.16}$$

$$\left\{\langle\hat{N}_{2,1}^n\rangle_t\right\}_{n=0}^{\infty}, \tag{3.17}$$

for any function $h(t)$ and

$$\left\{\theta\langle\hat{N}_1^n\rangle_t - \langle\hat{I}^n\rangle_t - \theta\langle\hat{\Delta}^{n-1}\rangle_t\right\}_{n=0}^{\infty}, \tag{3.18}$$

for $h(t) = 1$. Analogous expressions can be obtained for the sets SII [Eq. (3.4)] and SIII [Eq. (3.5)]. It then becomes clear that the particle current between levels ($\langle\hat{F}^0\rangle$) is equal to the photon flux. For $n = 0$ we obtain the conservation of the population of the levels and for $n > 0$ a restriction for the correlations ($\langle\hat{O}^n\rangle$).

For the case of the Jaynes-Cummings model the invariants read

$$\left\{ \frac{\langle \hat{\sigma}^n \rangle_t}{2} + \frac{\langle \hat{G}^{n+1} \rangle_t}{n+1} + \frac{\langle \hat{G}^n \rangle_t}{2} \right\}_{n=0}^{\infty} , \qquad (3.19)$$

for any function of time $h(t)$ and

$$\left\{ \langle \hat{I}^n \rangle_t + \frac{\theta}{2} \langle \hat{\sigma}^n \rangle_t + \frac{\theta}{2} \langle \hat{G}^n \rangle_t \right\}_{n=0}^{\infty} , \qquad (3.20)$$

for $h(t) = 1$. Note that for $n = 0$ the above invariants reduce to those given in Ref. [33]. Now, in terms of the nth-order coherence function, Eqs. (3.19) and (3.20) read

$$\left\{ \frac{\langle \hat{\sigma} G^n \rangle_t}{2} + \frac{\langle \hat{G}^{n+1} \rangle_t}{n+1} + \frac{\langle \hat{G}^n \rangle_t}{2} \right\}_{n=0}^{\infty} , \qquad (3.21)$$

for any function of time $h(t)$ and

$$\left\{ \sum_{r=0}^{n} \frac{(-1)^{n-r} n!}{2^{n-r} r!} \langle \hat{E}^r \rangle_t + \frac{\theta}{2} \left(\langle \hat{\sigma} \hat{G}^n \rangle_t + \langle \hat{G}^n \rangle_t \right) \right\}_{n=0}^{\infty} , \qquad (3.22)$$

for $h(t) = 1$, $\langle \hat{E}^r \rangle_t = \langle (\hat{I}^0 \ \hat{G}^r + \hat{G}^r \hat{I}^0)/2 \rangle_t$. These invariants of motion, evaluated for the time-dependent Jaynes-Cummings model, are expressed in terms of the coherence functions [30] broadly used in quantum optics experiments [see Eqs. (3.21), (3.22)]. Equations (3.16)–(3.22) show that the mean value of the operators will not be independent, *giving a restriction to the choice of the IC*. It is crucial to use a formalism that allows a proper evaluation of this consistent set of initial conditions. In the following section we will use the MEP density matrix given in Eq. (2.1) to generate a CSIC and we will obtain invariants of motion in the Lagrange multipliers dual space.

IV. QUANTUM THERMODYNAMICS, THE DUAL λ SPACE AND INITIAL CONDITIONS

The MEP context has the advantage that the IC can be chosen not only in the space of Lagrange multipliers but also in the space of mean values if carefully done. These Lagrange multipliers are numbers that can be freely chosen, while the mean values cannot [see Eq. (3.16)]. As we have shown before [14] the λ *dual space conserves or possesses all the restrictions coming from the fact that we are dealing with a quantal system* (i.e., the noncommutativity of the operators). This allows us to work in the λ space, preserving the quantal nature of the system. In order to do so the partition function λ_0 should be evaluated, as we will see below. The proper diagonalization of the density matrix can be done if the Hamiltonian of the system is introduced as a relevant operator. This leads to a nonzero temperature density operator. Once the diagonalization is made, λ_0 and the CSIC for the mean values can be obtained using Eq. (2.8). In our formalism the lack of knowledge on the mean value of one operator is equivalent to setting its Lagrange multiplier equal to zero. In order to derive the thermodynamical density matrix we include the Hamiltonian in the set of relevant operators; then the temperature of the system can be defined as [28]

$$\beta = \frac{1}{T} = \left. \frac{\partial S}{\partial \langle \hat{H} \rangle} \right|_{\langle \hat{O}_j \rangle} . \qquad (4.1)$$

In the problem considered here two temperatures can be defined [14]: one related to the total system (i.e., the two-level system plus the field) and the other associated with the quantum field. The last one takes into account only the thermodynamics of the field, and appears automatically in the formalism via the Lagrange multiplier associated with $\hat{\Delta}$. Therefore, to derive a thermodynamical solution to the problem at hand, we write the density matrix including the Hamiltonian as a relevant operator. Thus, the statistical operator can be written as

$$\hat{\rho}(t) = \exp\Bigg[-\lambda_0 \hat{I} - \beta \hat{H} - \sum_{n=0}^{\infty} \Big(\lambda_1^n \hat{N}_1^n + \lambda_2^n \hat{N}_2^n + \lambda_3^n \hat{F}^n + \lambda_4^n \hat{I}^n + \lambda_5^n \hat{N}_{2,1}^n + \lambda_6^n \hat{\Delta}^n \Big) \Bigg] . \qquad (4.2)$$

The diagonalization of Eq. (4.2) can be performed noting that the relevant physical operators do not introduce matrix elements different from zero outside the 2×2 blocks defined by the Hamiltonian. This is a consequence of the rotating wave approximation, which also determines the nonappearance of the electric and magnetic field as relevant operators (a detailed discussion of this point and related topics will be treated in a separate article). Thus, diagonalizing and evaluating the trace of $\hat{\rho}(t)$ we arrive at the following expression for λ_0 in terms of the other Lagrange multipliers:

$$\lambda_0 = \ln \left\{ \sum_{r=1}^{\infty} e^{-K_{1,r}} 2 \cosh(K_{2,r}) + e^{-\beta E_1 - \lambda_1^0} + \sum_{r=0}^{\infty} e^{-K_{3,r}} + \sum_{r=0}^{\infty} e^{-K_{4,r}} \right\} , \qquad (4.3)$$

where

$$K_{1,r} = \frac{\beta[E_2 + E_1 + (2r-1)\omega]}{2} + \sum_{n=0}^{r} \left[\frac{\lambda_1^n}{2} \Pi_r^{n-1} + \frac{\lambda_2^n + (2r-n-1)\lambda_6^n}{2} \Pi_{r-1}^{n-1} \right] , \qquad (4.4a)$$

$$K_{2,r} = \sqrt{X_r^2 + Y_r^2 + Z_r^2} , \qquad (4.4b)$$

$$K_{3,r} = \beta r \omega + \sum_{n=0}^{r} \lambda_6^n \Pi_r^n , \qquad (4.4c)$$

$$K_{4,r} = \beta(E_2 + E_1 + r\omega) + \sum_{n=0}^{r} \Pi_r^{n-1} \left[\lambda_1^n + \lambda_2^n + \lambda_5^n + (r-n)\lambda_6^n \right] , \qquad (4.4d)$$

are *invariants of the motion* [this can be shown using Eq. (2.7)],

$$X_r = \sqrt{r}\gamma \left(\beta h(t) + \sum_{n=0}^{r} \lambda_4^n \Pi_{r-1}^{n-1} \right), \tag{4.5a}$$

$$Y_r = \sqrt{r}\gamma \sum_{n=0}^{r} \lambda_3^n \Pi_{r-1}^{n-1}, \tag{4.5b}$$

$$Z_r = \frac{-\beta\theta}{2} + \sum_{n=0}^{r} \left(\frac{\lambda_1^n}{2} \Pi_r^{n-1} - \frac{\lambda_2^n - (n+1)\lambda_6^n}{2} \Pi_{r-1}^{n-1} \right), \tag{4.5c}$$

and $\Pi_r^m \equiv \prod_{j=0}^{m}(r-j)$, $\Pi_r^{-1} \equiv 1$. We want to mention that $\{X_r, Y_r, Z_r\}$ can be considered as a generalization of the vector model of density matrix shown in Ref. [30]. As in the case considered in that reference, one component of the vector is related to the population of the levels while the others are related to the real and imaginary parts of the nondiagonal elements of the density operator (interaction energy and particle current). In this sense we can consider $K_{2,r}$ as the norm of a vector in $\mathbb{R}^3$, with components $\{X_r, Y_r, Z_r\}$. So, these r-dependent spheres can be thought of as an extension in the dual space of Lagrange multipliers of a sort of *quantized version of the Bloch sphere.* Notice that λ_0 is the partition function of the system. This means that all the thermodynamical quantities can be computed [28].

Applying Eqs. (2.8) and (4.3) the CSIC is obtained. For example, the initial mean value of the population of the level one and its correlations with the field reads

$$\langle \hat{N}_1^n \rangle_0 = e^{-\lambda_0} \left\{ e^{-\beta E_1 - \lambda_1^0} \delta_{n,0} + \sum_{r=1}^{\infty} \Pi_r^{n-1} \left[e^{-K_{1,r}} \left(\cosh(K_{2,r}) - \frac{Z_r}{K_{2,r}} \sinh(K_{2,r}) \right) + \sum_{r=0}^{\infty} \Pi_r^{n-1} e^{-K_{4,r}} \right] \right\}, \tag{4.6}$$

where δ is the Kronecker function. In order to compare with the IC usually shown in literature, we study some special sets of CSIC. Notice that in our formalism a CSIC implies having the partition function λ_0 of the problem.

A. Noninteracting IC

Usually, the initial knowledge is restricted to the population of the levels and the distribution of photons of the field. Thus, in the Lagrange multipliers dual space this IC reads: $\lambda_1^0 \neq 0$, $\lambda_2^0 \neq 0$, $\lambda_6^n \neq 0$ with $n \geq 0$, taking the remaining Lagrange multipliers equal to zero (complete lack of knowledge). This IC will be called *noninteracting.* Thus, we obtain

$$\langle \hat{N}_1^n \rangle_0 = \langle \hat{N}_1^0 \rangle_0 \langle \hat{\Delta}^{n-1} \rangle_0, \tag{4.7a}$$
$$\langle \hat{N}_2^n \rangle_0 = \langle \hat{N}_2^0 \rangle_0 \langle \hat{\Delta}^{n-1} \rangle_0, \tag{4.7b}$$
$$\langle \hat{N}_{2,1}^n \rangle_0 = 0, \tag{4.7c}$$
$$\langle \hat{F}^n \rangle_0 = 0, \tag{4.7d}$$
$$\langle \hat{I}^n \rangle_0 = 0, \tag{4.7e}$$
$$\langle \hat{\Delta}^n \rangle_0 = \frac{\sum_{r=0}^{\infty} \Pi_r^n \exp\left(-\sum_{j=0}^{r} \Pi_r^j \lambda_6^j \right)}{\sum_{r=0}^{\infty} \exp\left(-\sum_{j=0}^{r} \Pi_r^j \lambda_6^j \right)}, \tag{4.7f}$$

where

$$\langle \hat{N}_1^0 \rangle_0 = \frac{e^{-\lambda_1^0} + e^{-\lambda_1^0 - \lambda_2^0}}{1 + e^{-\lambda_1^0} + e^{-\lambda_2^0} + e^{-\lambda_1^0 - \lambda_2^0}}, \tag{4.8a}$$
$$\langle \hat{N}_2^0 \rangle_0 = \frac{e^{-\lambda_2^0} + e^{-\lambda_1^0 - \lambda_2^0}}{1 + e^{-\lambda_1^0} + e^{-\lambda_2^0} + e^{-\lambda_1^0 - \lambda_2^0}}. \tag{4.8b}$$

Notice that for the special case considered here, the correlations between the population of the levels and the field are initially decoupled for any initial distribution of the field (i.e., $\beta = 0$, $\lambda_3^n = 0$, $\lambda_4^n = 0$ $\forall n \geq 0$).

B. Pure state IC

Let the field be initially in an eigenstate of the Hamiltonian. This pure state case is the simplest IC that can be studied. When the field is initially in a state $|m\rangle$, Eqs. (4.7) read

$$\langle \hat{N}_1^n \rangle_0 = \langle \hat{N}_1^0 \rangle_0 \langle \hat{\Delta}^{n-1} \rangle_0, \tag{4.9a}$$
$$\langle \hat{N}_2^n \rangle_0 = \langle \hat{N}_2^0 \rangle_0 \langle \hat{\Delta}^{n-1} \rangle_0, \tag{4.9b}$$
$$\langle \hat{N}_{2,1}^n \rangle_0 = 0, \tag{4.9c}$$
$$\langle \hat{F}^n \rangle_0 = 0, \tag{4.9d}$$
$$\langle \hat{I}^n \rangle_0 = 0, \tag{4.9e}$$
$$\langle \hat{\Delta}^n \rangle_0 = \Pi_m^n, \tag{4.9f}$$

where $\langle \hat{N}_1^0 \rangle$ and $\langle \hat{N}_2^0 \rangle$ are defined as in Eqs. (4.8). Note that when $n > m$ $\langle \hat{\Delta}^n \rangle_0 = 0$. Thus, for this IC only a finite number of moments of the population number of the field are initially different from zero. The mean value of the population number is m and

$$\langle (\hat{a}^\dagger \hat{a})^2 \rangle_0 - \langle \hat{a}^\dagger \hat{a} \rangle_0^2 = 0. \tag{4.10}$$

C. Coherent state IC

Another IC broadly used in the literature is the coherent state. For instance, collapse and revivals were first observed for this IC. If the field is initially in a coherent state $|\alpha\rangle$, Eqs. (4.7) read

$$\langle \hat{N}_1^n \rangle_0 = \langle \hat{N}_1^0 \rangle_0 \langle \hat{\Delta}^{n-1} \rangle_0, \tag{4.11a}$$
$$\langle \hat{N}_2^n \rangle_0 = \langle \hat{N}_2^0 \rangle_0 \langle \hat{\Delta}^{n-1} \rangle_0, \tag{4.11b}$$
$$\langle \hat{N}_{2,1}^n \rangle_0 = 0, \tag{4.11c}$$
$$\langle \hat{F}^n \rangle_0 = 0, \tag{4.11d}$$
$$\langle \hat{I}^n \rangle_0 = 0, \tag{4.11e}$$
$$\langle \hat{\Delta}^n \rangle_0 = (\langle \hat{\Delta}^0 \rangle_0)^{n+1} = |\alpha|^{2(n+1)}, \tag{4.11f}$$

where $\langle \hat{N}_1^0 \rangle$ and $\langle \hat{N}_2^0 \rangle$ are defined as in Eqs. (4.8). Thus, for this IC the mean value of the population number is $|\alpha|^2$ and

$$\langle(\hat{a}^\dagger\hat{a})^2\rangle_0 - \langle\hat{a}^\dagger\hat{a}\rangle_0^2 = |\alpha|^2. \quad (4.12)$$

Let us observe the fact that the mean value of the moments of the number of photons of the field increase very fast as n grows. This tells us that as we increase the number of photons of the field, the system becomes more and more correlated. This fact will be reflected in the numerical study of the evolution Eqs. (3.15).

D. Thermal state IC

Now, let us consider the case of having the field initially in a thermal state. This IC is a *particular* case of Eqs. (4.7) and has been studied by Knight and co-workers (see for instance Refs. [29,31,32]). In the Lagrange multipliers dual space this IC reads $\lambda_1^0 \neq 0$, $\lambda_2^0 \neq 0$, $\lambda_6^0 \neq 0$ and the other Lagrange multipliers are equal to zero. This peculiar selection of λ gives automatically the IC for a thermal state of the field (see Ref. [33]). The initial mean value of $\hat{\Delta}^n$ reads

$$\langle\hat{\Delta}^n\rangle_0 = \frac{(n+1)!}{(e^{\lambda_6^0(0)}-1)^{n+1}} = (n+1)!(\langle\hat{\Delta}^0\rangle_0)^{n+1}. \quad (4.13)$$

The other IC are given by Eqs. (4.7). Notice that for $n=0$ we have

$$\langle\hat{\Delta}^0\rangle_0 = \langle\hat{a}^\dagger\hat{a}\rangle_0 = \frac{1}{(e^{\lambda_6^0(0)}-1)}. \quad (4.14)$$

From this expression we see that $\lambda_6^0(0)$ is equal to $\beta^*\omega$, where β^* is the temperature associated only with the field. It is clear that $\lambda_6^0(0)$ should be a positive number since the temperature is a positive number. Summarizing, the solution for the usually called *thermal state* results as a *particular selection of the IC for the values of λ in the λ dual space.* Looking at Eqs. (4.11f) and (4.13) we immediately see that if in both cases the initial population of the field is $\langle\hat{\Delta}^0\rangle_0$, we obtain

$$\langle\hat{\Delta}^n\rangle_{\rm th} = (n+1)!\langle\hat{\Delta}^n\rangle_{\rm coh}. \quad (4.15)$$

This relation shows that for a thermal state, the values of the correlation are even greater than in the coherent case. Therefore, a larger number of equations are needed for larger values of $\langle\hat{\Delta}^0\rangle_0$ when this case is numerically studied.

V. GENERALIZED TIME-INDEPENDENT JAYNES-CUMMINGS HAMILTONIAN

Now, we study the time-independent Hamiltonian case which leads to the standard Jaynes-Cummings Hamiltonian. The Hamiltonian reads

$$\hat{H} = E_1\hat{b}_1^\dagger\hat{b}_1 + E_2\hat{b}_2^\dagger\hat{b}_2 + \omega\hat{a}^\dagger\hat{a} + \gamma\hat{a}\hat{b}_1\hat{b}_2^\dagger + \gamma^*\hat{b}_2\hat{b}_1^\dagger\hat{a}^\dagger. \quad (5.1)$$

For this case, Eqs. (3.15) can be explicitly solved using a series expansion method. If the Hamiltonian is time independent and $\langle\hat{O}\rangle_t$ is the mean value of a quantum operator, its evolution can be written in power series as follows:

$$\langle\hat{O}\rangle_t = \langle\hat{O}\rangle_0 + \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{i\hbar}\right)^n \langle[\cdots[\hat{O},\hat{H}],\ldots,\hat{H}]\rangle_0, \quad (5.2)$$

where each term has n commutators. Using Eq. (2.5), $[\cdots[\hat{O},\hat{H}],\ldots,\hat{H}]$ can be expressed in terms of the set of relevant operators and the matrix G. To fix ideas let us choose $\hat{N}_1^0$ as $\hat{O}$. If one introduces the quantized generalized Rabi flopping frequency [30], $\Omega_n^2 \equiv \theta^2 + (n+1)\epsilon^2 \equiv \Omega_0^2 + n\epsilon^2$, $\epsilon^2 \equiv 4|\gamma|^2$, and $\Theta_{nk} \equiv \Omega_n/\Omega_k$, the exact solution for $\langle\hat{N}_1\rangle_t$ becomes

$$\begin{aligned}\langle\hat{N}_1\rangle_t = {}& \langle\hat{N}_1\rangle_0 + \frac{\langle\hat{F}\rangle_0}{\Omega_0}S_0(t) + \left(\frac{\langle\hat{\Lambda}^0\rangle_0}{\Omega_0^2} - \frac{\epsilon^2\langle\hat{N}_2\rangle_0}{2\Omega_0^2}\right)C_0(t) + \sum_{n=1}^{\infty}\left(\frac{\langle\hat{\Lambda}^n\rangle_0}{\Omega_n^2} + \frac{\theta^2\langle\hat{N}_2^n\rangle_0}{2\Omega_n^2}\right)\\ &\times\sum_{k=0}^{n-1}b_{n,k}\left[C_n(t) - \Theta_{nk}^{2n}C_k(t)\right] - \sum_{n=1}^{\infty}\frac{\langle\hat{N}_2^n\rangle_0}{2}\sum_{k=0}^{n-1}b_{n,k}\left[C_n(t) - \Theta_{nk}^{2n-2}C_k(t)\right]\\ &+\sum_{n=1}^{\infty}\frac{\langle\hat{F}^n\rangle_0}{\Omega_n}\sum_{k=0}^{n-1}b_{n,k}\left[S_n(t) - \Theta_{nk}^{2n-1}S_k(t)\right], \end{aligned} \quad (5.3)$$

where $\langle\hat{\Lambda}^k\rangle_0 = \epsilon^2\langle\hat{N}_1^{k+1}\rangle_0/2 + \theta\langle\hat{I}^k\rangle_0 + \epsilon^2(k+1)\langle\hat{N}_{2,1}^k\rangle_0/2$, $b_{n,k} \equiv a_{n,k}\Theta_{kn}^{2n-2}$, $C_j(t) \equiv \cos(\Omega_j t) - 1$, $S_j(t) \equiv \sin(\Omega_j t)$, and $a_{n,k} = (-1)^{n+k+1}/(n-k)!k!$. A remarkable property of Eq. (5.3) is that the first nonvanishing term of the correlation $\langle\hat{O}_i^n\rangle_0$ in this solution is proportional to t^{2n}, t^{2n+1}, or t^{2n+2}, depending on the different operators $\hat{O}_i$. This can be seen if one makes a Taylor's expansion of $\langle\hat{O}_i^n\rangle_t$ and uses Eqs. (3.15) together with the fact that the $a_{n,k}$ satisfy the following Vandermonde-like system of equations [34,35]

$$\sum_{k=0}^{n-1}a_{n,k}(n-k)k^i = \delta_{i,n-1}, \qquad i = 0,\ldots,n-1. \quad (5.4)$$

Thus, up to a given time, there will be only a finite number of correlations that will contribute substantially to the solution (5.3). Similar expressions can be obtained for all the operators. Using that

$$\sum_{k=0}^{n-1} a_{n,k}\Theta_{kn}^{2n-2} = -a_{n,n}, \tag{5.5}$$

Eq. (5.3) simplifies to

$$\begin{aligned}\langle \hat{N}_1\rangle_t = \langle \hat{N}_1\rangle_0 &- \sum_{n=0}^{\infty}\langle \hat{\Lambda}^n\rangle_0 \sum_{k=0}^{n} a_{n,k}\frac{C_k(t)}{\Omega_k^2} \\ &+ \sum_{n=0}^{\infty}\frac{\epsilon^2\langle \hat{N}_2^n\rangle_0}{2}\sum_{k=0}^{n} a_{n,k}(k+1)\frac{C_k(t)}{\Omega_k^2} \\ &- \sum_{n=1}^{\infty}\langle \hat{F}^n\rangle_0 \sum_{k=0}^{n} a_{n,k}\frac{S_k(t)}{\Omega_k}. \end{aligned}\tag{5.6}$$

Now, and in order to see how this expression reduces to the well known ones [30] we rewrite Eq. (5.3) for the initial conditions considered in the preceding section. We specialize our results for the special case in which the correlations between the population of the levels and the field are initially decoupled for any initial distribution of the field (i.e., $\beta = 0$, $\lambda_3^n = 0$, $\lambda_4^n = 0\ \forall n \geq 0$) and resonance [i.e., $\theta = 0$ and $\Theta_{nk} = \sqrt{(n+1)/(k+1)}$]. We obtain

$$\begin{aligned}\langle \hat{N}_1^0\rangle_t = \langle \hat{N}_1^0\rangle_0 + \frac{1}{2}\sum_{n=0}^{\infty}\Bigg(&\langle \hat{N}_1^{n+1}\rangle_0 \sum_{k=1}^{n-1}\frac{a_{n,k}(k+1)^{n-1}}{(n+1)^n} \\ &- \sum_{k=n+1}^{\infty}\langle \hat{N}_1^{k+1}\rangle_0\frac{a_{k,n}}{(n+1)}\Bigg) C_n(t),\end{aligned}\tag{5.7}$$

where we have used that

$$\sum_{n=0}^{\infty}\langle \hat{N}_1^{n+1}\rangle_0 \sum_{k=0}^{n-1}\frac{a_{n,k}}{(k+1)}C_k(t) = \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\langle \hat{N}_1^{k+1}\rangle_0\frac{a_{k,n}}{(n+1)}C_n(t). \tag{5.8}$$

Note that in general the evolution of the population inversion is expressed as an infinite sum of cosines with frequencies proportional to $\sqrt{n}$, weighted by the density of photons of the quantum field. In our case, as it was expected, we have the same cosines but weighted by coefficients proportional to the mean values of the correlation functions defined by Eqs. (3.3). Now, we rewrite Eq. (5.7) for the special cases considered in Sec. IV.

A. Pure state

If the field is initially prepared in a number state $|m\rangle$ and we have one particle in level one, the time evolution of the population of level one is described by the well known Rabi solution given by

$$\langle \hat{N}_1^0\rangle_t = \tfrac{1}{2}[1+\cos(2|\gamma|\sqrt{m}t)], \tag{5.9}$$

which has been obtained replacing Eqs. (4.9), in (5.7) and using Eq. (5.13).

B. Coherent state

If the initial state of the field is a coherent one, Eq. (5.7) reads

$$\langle \hat{N}_1^0\rangle_t = 1 + \frac{1}{2}\sum_{k=0}^{\infty}\frac{\langle\hat{\Delta}^0\rangle_0^{n+1}e^{-\langle\hat{\Delta}^0\rangle_0}}{(n+1)!}C_n(t), \tag{5.10}$$

where Eqs. (4.11) and

$$\sum_{k=n+1}^{\infty}\frac{(-1)^k\langle\hat{\Delta}^0\rangle_0^{k+1}}{(k-n)!} = (-\langle\hat{\Delta}^0\rangle_0)^{n+1}\left(1-e^{-\langle\hat{\Delta}^0\rangle_0^{n+1}}\right) \tag{5.11}$$

were used. As can be seen, the cosines are weighted by a Poisson distribution of photons. Now, performing simple algebra we obtain the formula for the population of the first level as follows:

$$\langle \hat{N}_1^0\rangle_t = \frac{1}{2}\left(1+\sum_{n=0}^{\infty}\frac{\langle\hat{\Delta}^0\rangle_0^{n}e^{-\langle\hat{\Delta}^0\rangle_0}}{n!}\cos(2|\gamma|\sqrt{n}t)\right). \tag{5.12}$$

It is well known that the Poisson spread of Rabi frequencies dephases, or collapses, the Rabi oscillations with a time independent of n. The Rabi oscillations partly rephase, or revive. Collapse and revival are a fundamental consequence of the discreteness of the quantized field mode and they have no classical counterpart. For an extended discussion of these and related topics, see, for instance, Refs. [10,29].

C. Thermal state

We now consider the special case of having one particle in level one and the field in a thermal state. The distribution of photons associated with this IC is the Bose-Einstein distribution. Using Eqs. (4.7), (4.13), (5.7),

$$\sum_{k=0}^{n-1}\frac{(-1)^{k+n+1}n!}{(n-k)!k!}\left(\frac{k+1}{n+1}\right)^{n-1} = 1 \tag{5.13}$$

and

$$\sum_{k=n+1}^{\infty}\frac{\langle\hat{\Delta}^0\rangle_0^{k+1}(-1)^{k+n+1}(k+1)!}{(k-n)!(n+1)!} = (-1)^{2n+1}\langle\hat{\Delta}^0\rangle_0^{n+1}\left(\frac{1}{(1+\langle\hat{\Delta}^0\rangle_0)^{n+2}}-1\right), \tag{5.14}$$

we obtain

$$\langle \hat{N}_1^0 \rangle_t = \frac{1}{2}\left(1 + \sum_{k=0}^{\infty} \frac{\langle \hat{\Delta}^0 \rangle_0^n}{(1+\langle \hat{\Delta}^0 \rangle_0)^{n+1}} \cos(2|\gamma|\sqrt{n}t)\right). \tag{5.15}$$

Note that we have a distribution of Rabi frequencies. Knight and Radmore found that also for this case the Rabi oscillations collapse and revive as a consequence of the discreteness of the quantum field (for a detailed analysis of this effect we refer the reader to Ref. [32]).

VI. NUMERICAL RESULTS

In this section we are going to study the evolution of the population inversion $\langle\hat{\sigma}\rangle_t$ and the second-order coherence function $g^2(t)$ for a linear and exponential coupling and with the field initially in a coherent and thermal state. The numerical results were obtained by choosing an interval of time and neglecting those correlations that do not contribute to the solution for the given time interval. Remember that an analogous approximation should be done, when calculating $\langle\hat{\sigma}\rangle_t$ from the series solution. That means that from the numerical point of view both procedures give an exact numerical solution up to a given time since neither the series nor the system of equations have, in general, a closed analytical solution.

In Fig. 1 we see how the different order of the correlations enters into the evolution of $\langle\hat{\sigma}\rangle_t$. The field is initially in a coherent state with $\langle\hat{\Delta}^0\rangle_0 = 10$, the two-level system is initially in the excited state and we use noninteracting IC (see Sec. IV A). Thus, $\langle\hat{\sigma}^j\rangle_0 = 10^j$, $\langle\hat{\Delta}^j\rangle_0 = 10^{j+1}$, $\langle\hat{N}_{2,1}^j\rangle_0 = \langle\hat{F}^j\rangle_0 = \langle\hat{I}^j\rangle_0 = 0$ for $0 \le j \le n$. We also assume that the system is in resonance (i.e., $\theta = 0$). In all cases we plot in dashed line the evolution for $\langle\hat{\sigma}\rangle_t$ for $n = 80$ (n is the higher correlation included in the truncated solution). For this we have numerically proven that it is not necessary to include higher-order correlations and thus we will say that the solution is exact for the chosen adimensional time interval. It is also for $n = 80$ that we have obtained a complete agreement with the evolution obtained from the series solution. Figure 1(a) shows in solid line $\langle\hat{\sigma}\rangle_t$ for $n = 57$. We see that for times smaller than $10/\gamma$ both curves coincided. For later times they start to differ and, moreover, for times greater than $30/\gamma$ the evolution goes beyond the bounds $[-1, 1]$. Figure 1(b) shows the improvement when we increase the order in one unit ($n = 58$). Now, the curve is bounded between $[-1, 1]$ and there is a better agreement between the solid and dashed curves for times larger than $10/\gamma$. In Fig. 1(c) we increase by two the number of correlations ($n = 60$). We observe that there is a better agreement between both curves, but a difference remains for times greater than $25/\gamma$. In Fig. 1(d) we plot $\langle\hat{\sigma}\rangle_t$ for $n = 70$. We now see that both curves are in complete agreement and, therefore, it is not necessary to include more correlations.

For the time-dependent case, analytical expressions have been found, to the best of our knowledge, in few cases [14,22–27]. Recently, Prants and Yacoupova [25]

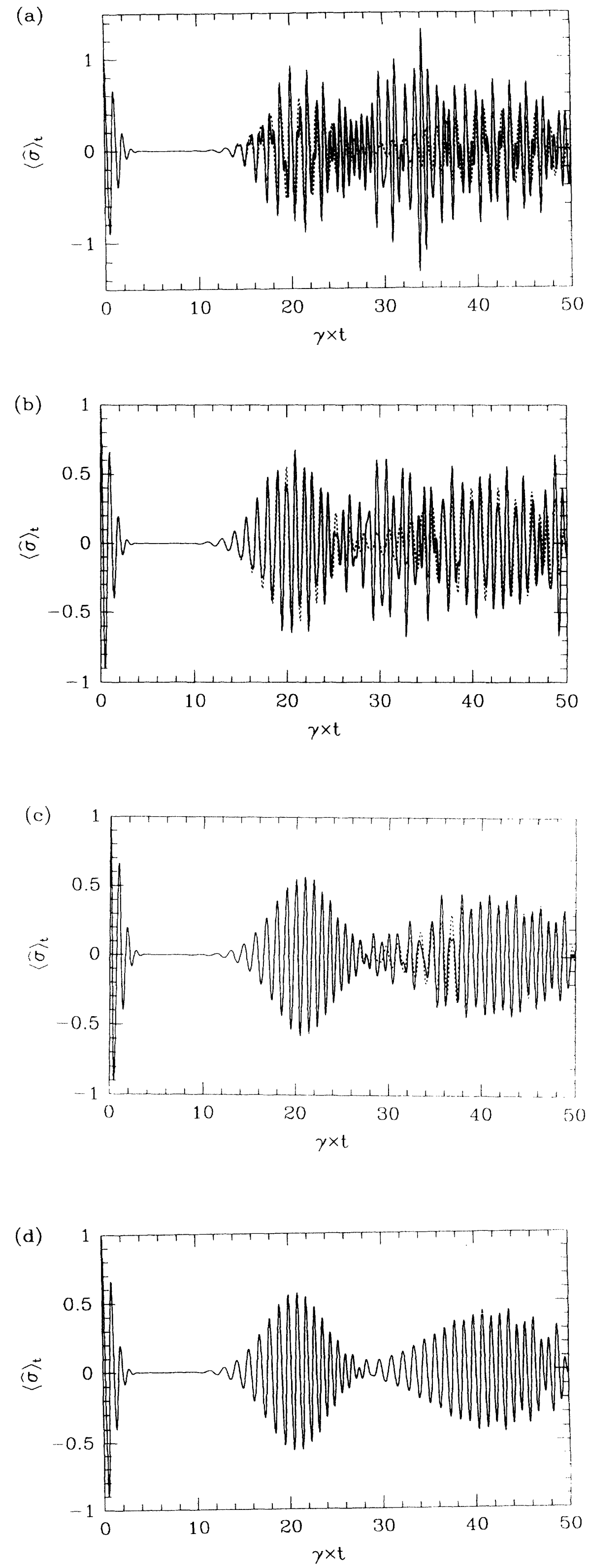

FIG. 1. Temporal evolution of $\langle\hat{\sigma}\rangle_t$ for different orders of n. The solid line corresponds to (a) $n = 57$, (b) $n = 58$, (c) $n = 60$, (d) $n = 70$. In all cases $h(t) = 1$, $\theta = 0$, the field is initially in a coherent state with $\langle\hat{\Delta}^0\rangle_0 = 10$, the two-level system is in the excited state, we use noninteracting IC, and for the dashed line $n = 80$.

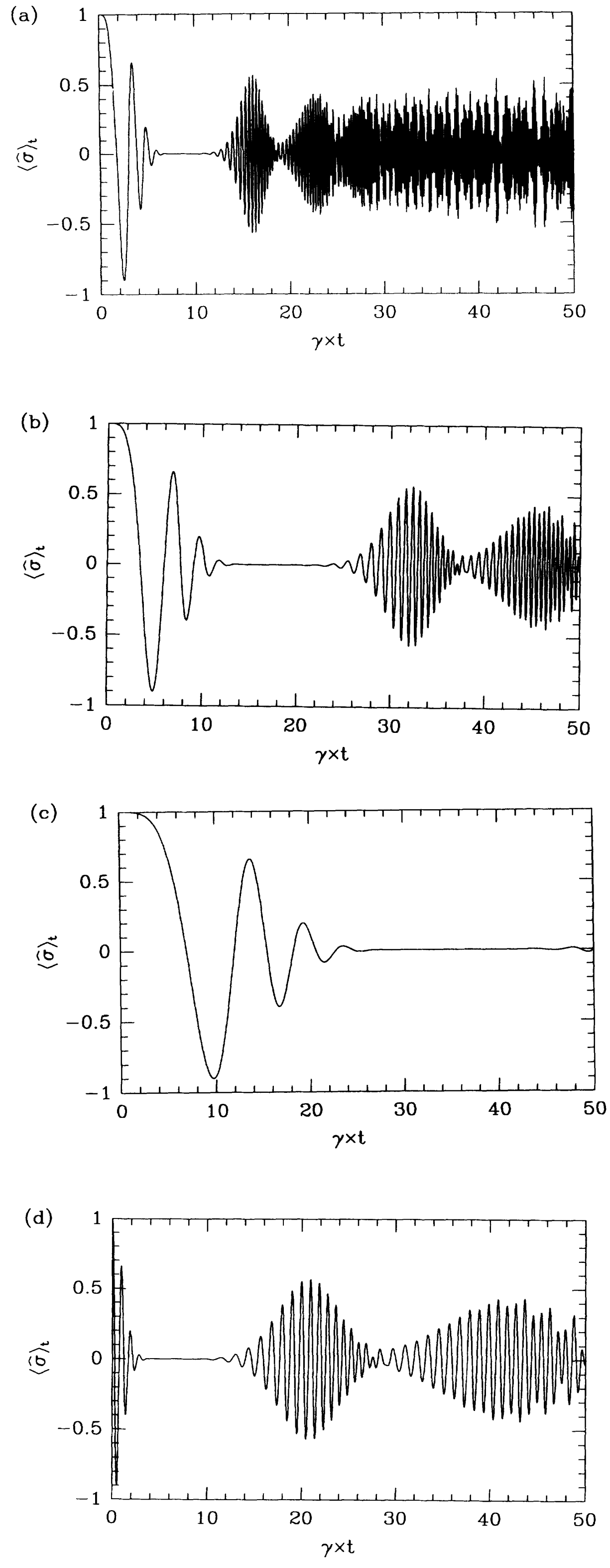

FIG. 2. Temporal evolution of $\langle\hat{\sigma}\rangle_t$ with a linear time-dependent coupling: (a) $c = 8.0$, (b) $c = 2.0$, (c) $c = 0.5$. Curve (d) represents $\langle\hat{\sigma}\rangle_t$ for the time-independent case $[h(t) = 1]$. In all the cases $\tau\gamma = 50$, $n = 80$, $\theta = 0$, the field is initially in a coherent state with $\langle\hat{\Delta}^0\rangle_0 = 10$, the two-level system is in the excited state, and we use noninteracting IC.

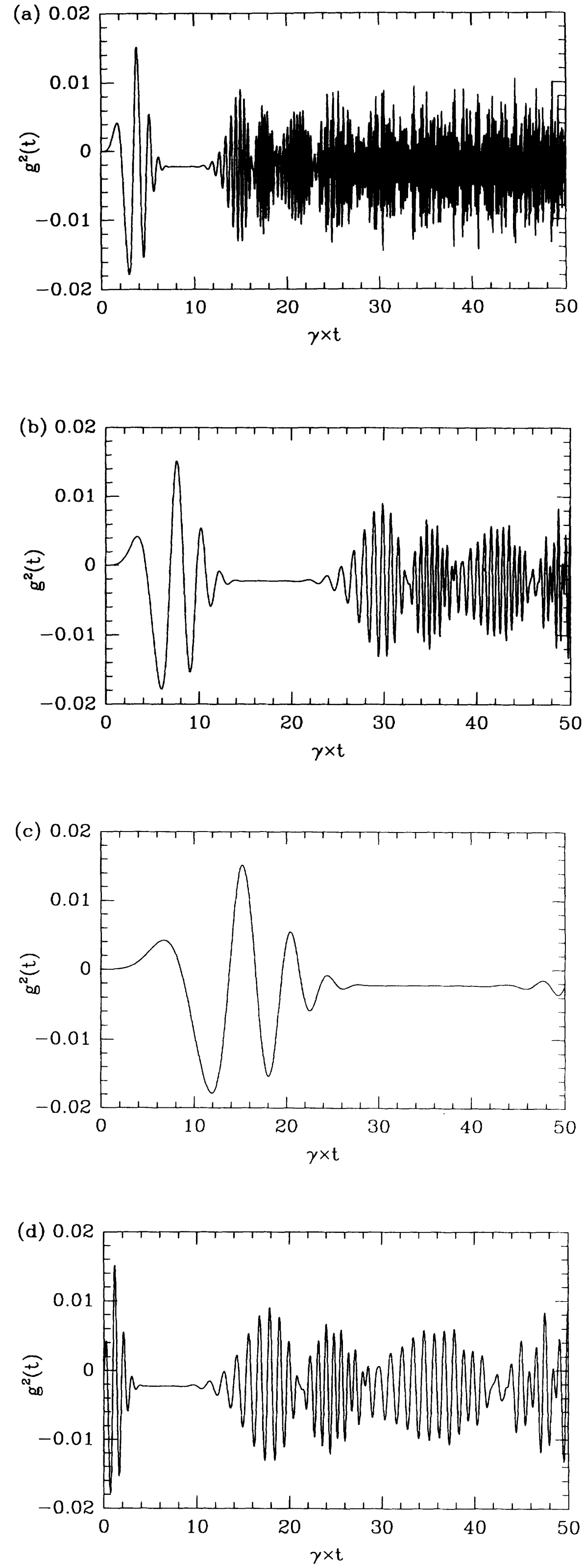

FIG. 3. Temporal evolution of $g^2(t)$ with a linear time-dependent coupling: (a) $c = 8.0$, (b) $c = 2.0$, (c) $c = 0.5$. Curve (d) shows $g^2(t)$ for the time-independent case $[h(t) = 1]$. In all the cases $\tau\gamma = 50$, $n = 80$, $\theta = 0$, the field is initially in a coherent state with $\langle\hat{\Delta}^0\rangle_0 = 10$, the two-level system is in the excited state, and we use noninteracting IC.

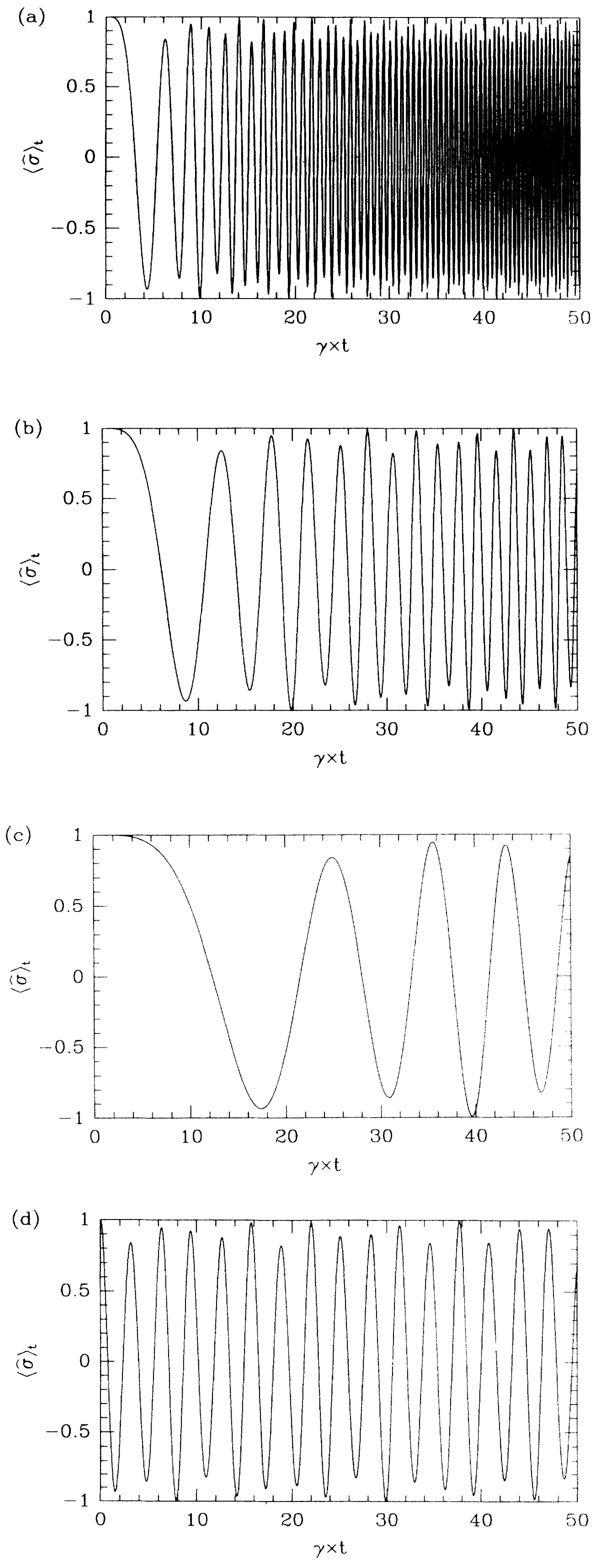

FIG. 4. Temporal evolution of $\langle\hat{\sigma}\rangle_t$ with a linear time-dependent coupling: (a) $c = 8.0$, (b) $c = 2.0$, (c) $c = 0.5$. Curve (d) represents $\langle\hat{\sigma}\rangle_t$ for the time-independent case $[h(t) = 1]$. In all the cases $\tau\gamma = 50$, $n = 20$, $\theta = 0$, the field is initially in a thermal state with $\langle\hat{\Delta}^0\rangle_0 = 0.1$, the two-level system is in the excited state, and we use noninteracting IC.

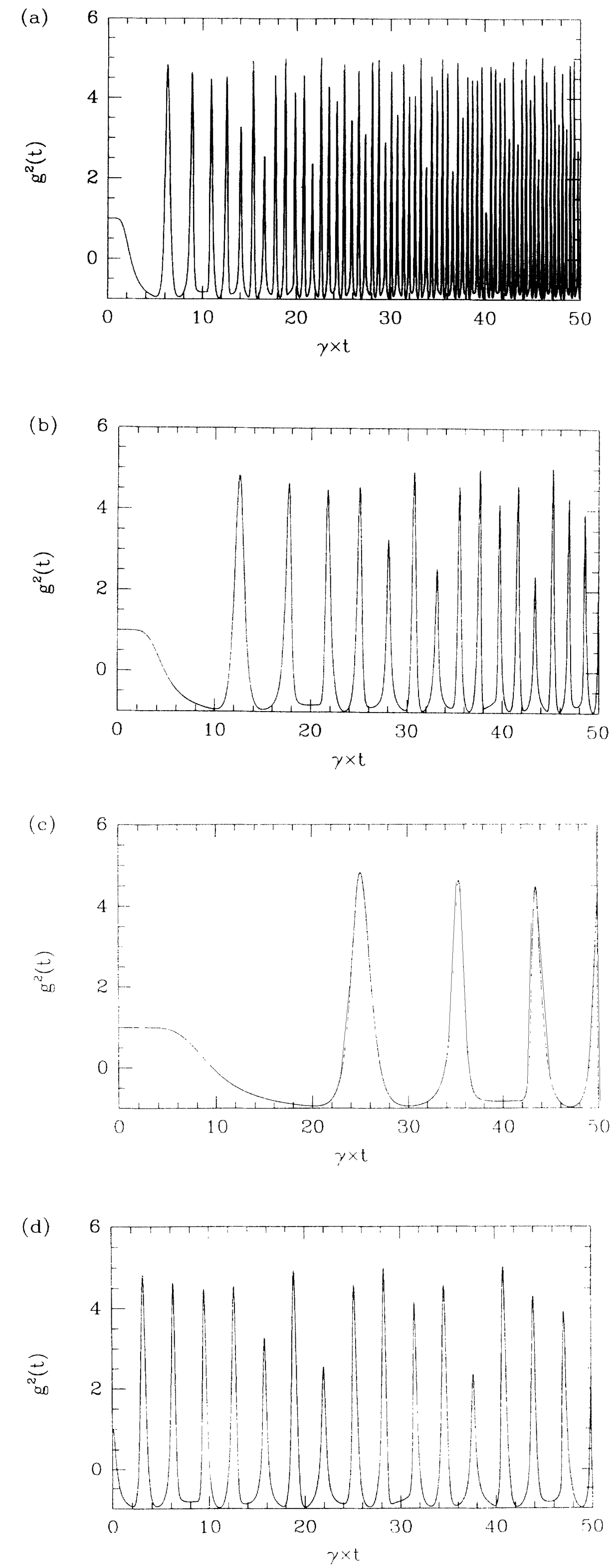

FIG. 5. Temporal evolution of $g^2(t)$ with a linear time-dependent coupling: (a) $c = 8.0$, (b) $c = 2.0$, (c) $c = 0.5$. Curve (d) represents $g^2(t)$ for the time-independent case $[h(t) = 1]$. In all the cases $\tau\gamma = 50$, $n = 20$, $\theta = 0$, the field is initially in a thermal state with $\langle\hat{\Delta}^0\rangle_0 = 0.1$, the two-level system is in the excited state, and we use noninteracting IC.

found analytical expression for an exponential modulation of the coupling. Joshi and Lawande [26] have described the dynamics of $\langle\hat{\sigma}\rangle_t$ and $g^2(t)$ for a linear sweep. We will show that our results, obtained from the system Eq. (3.15) agree with the numerical results of Refs. [25,26]. Moreover, we also calculate the dynamical evolution for the case of having the field in a thermal state.

We start analyzing the linear case. Thus $h(t)$ takes the form

$$h(t) = \begin{cases} ct/\tau & \text{for } 0 \leq t \leq \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Figures 2(a–c) and 3(a–c) show the evolution of $\langle\hat{\sigma}\rangle_t$ and $g^2(t)$, respectively, for the linear sweep [(a) $c = 8.0$, (b) $c = 2.0$, (c) $c = 0.5$]. In Figs. 2(d) and 3(d) the time-independent case is depicted. We have used the same IC as that in Fig. 1. We have chosen $\tau\gamma = 50$, $n = 80$ for all the cases and resonance ($\theta = 0$). We see that our results are in complete agreement with those of Ref. [26]. In Fig. 2(a) we see the collapse and revivals of $\langle\hat{\sigma}\rangle_t$ expected when the field is initially in a coherent state. Figure 3 shows the antibunching or sub-Poissonian distribution which has no classical counterpart.

In Figs. 4 and 5 we study the time evolution of $\langle\hat{\sigma}\rangle_t$

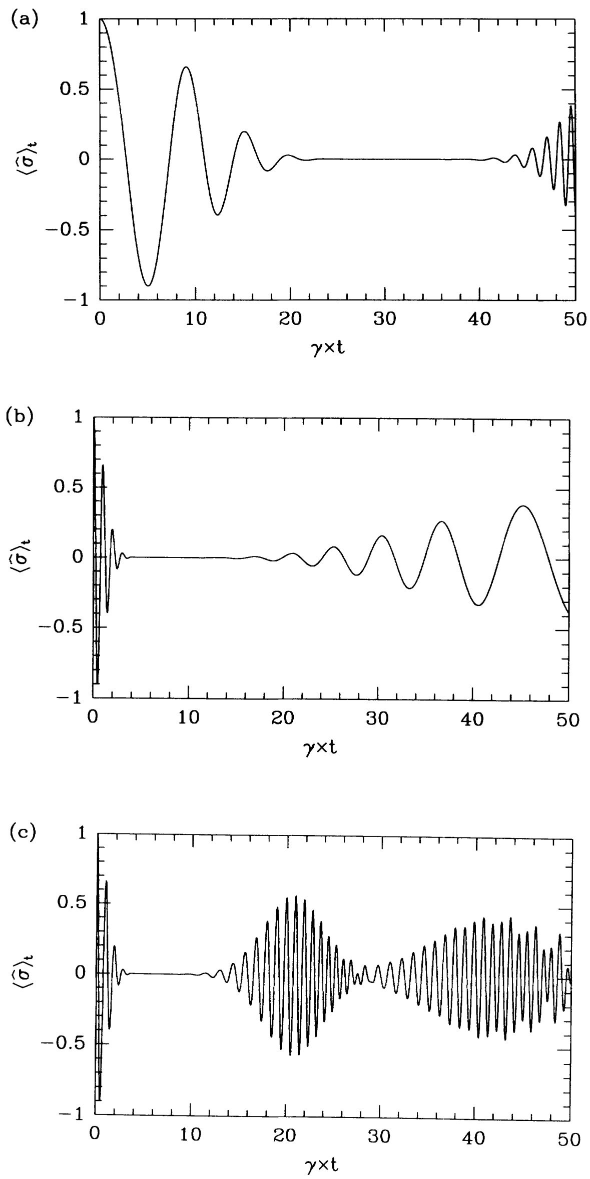

FIG. 6. Temporal evolution $\langle\hat{\sigma}\rangle_t$ with an exponential time-dependent coupling: (a) switch on, $K/\gamma = -0.05$, $t_0 = 0$; (b) switch off, $K/\gamma = 0.05$, $t_0 = \tau$. Curve (c) represents $\langle\hat{\sigma}\rangle_t$ for the time-independent case [$h(t) = 1$]. In all the cases $\tau\gamma = 50$, $n = 80$, $\theta = 0$, the field is initially in a coherent state with $\langle\hat{\Delta}^0\rangle_0 = 10$, the two-level system is in the excited state, and we use noninteracting IC.

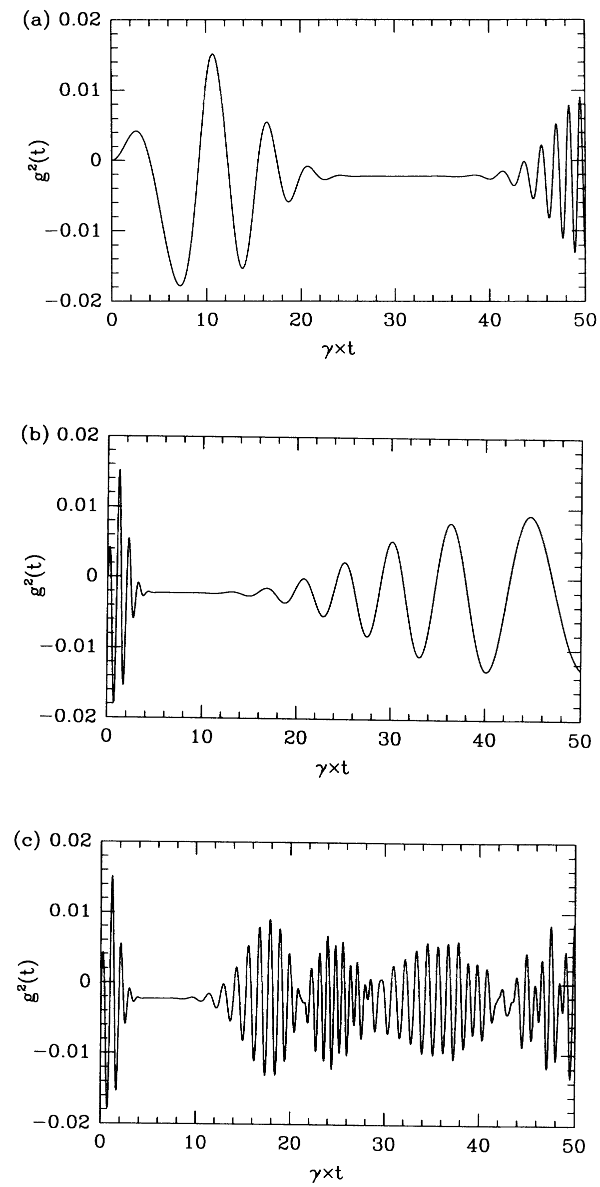

FIG. 7. Temporal evolution of $g^2(t)$ with an exponential time-dependent coupling: (a) switch on, $K/\gamma = -0.05$, $t_0 = 0$; (b) switch off, $K/\gamma = 0.05$, $t_0 = \tau$. Curve (c) represents $g^2(t)$ for the time-independent case [$h(t) = 1$]. In all the cases $\tau\gamma = 50$, $n = 80$, $\theta = 0$, the field is initially in a coherent state with $\langle\hat{\Delta}^0\rangle_0 = 10$, the two-level system is in the excited state, and we use noninteracting IC.

and $g^2(t)$ for the linear sweep, with noninteracting IC but the field is initially in a thermal state. For this case the IC are: $\langle\hat{\sigma}^j\rangle_0 = 0.1^j j!$, $\langle\hat{\Delta}^j\rangle_0 = 0.1^{j+1}(j+1)!$, $\langle\hat{N}^j_{2,1}\rangle_0 = \langle\hat{F}^j\rangle_0 = \langle\hat{I}^j\rangle_0 = 0$ for $0 \le j \le n$. We took $\tau\gamma = 50$, $n = 20$ and resonance. We observe the same change in frequency as the one obtained for the coherent case. As can be seen, the frequency decreases as we decrease the values of c. Note that in all cases the mean features of the curves compared with the time-independent case are the same. That means that as we increase or decrease the intensity of the interaction we are effectively changing the value of the Rabi frequencies.

We study now the case of an exponential modulation of the two-level system-field coupling. Thus

$$h(t) = \begin{cases} \exp(-K(t-t_0) - |K|\tau) & \text{for } 0 \le t \le \tau, \\ 0 & \text{otherwise.} \end{cases} \qquad (6.2)$$

Choosing a negative (positive) value for K we can simulate the switch on (switch off) process. In Figs. 6(a) and 6(b) we plot $\langle\hat{\sigma}\rangle_t$ with $n = 80$, $\theta = 0$, $\tau\gamma = 50$, $|K|/\gamma = 0.05$. We have used the same IC as that in Fig. 1. We observe that during the switch on process [Fig. 6(a)] the period of the first oscillation before the collapse increases with respect to the time-independent

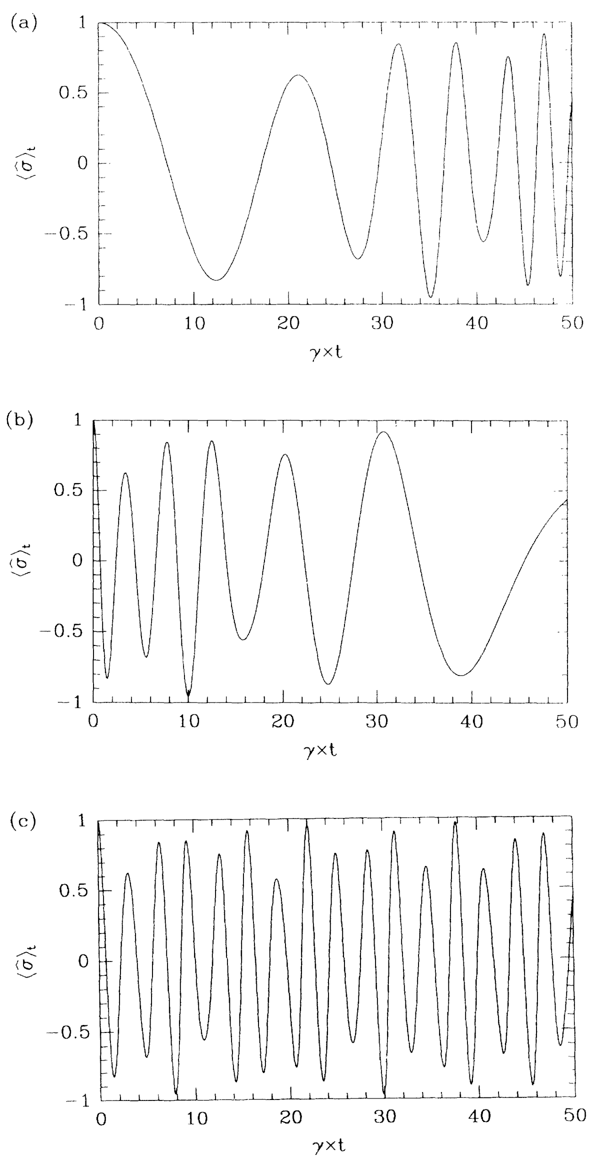

FIG. 8. Temporal evolution of $\langle\hat{\sigma}\rangle_t$ with an exponential time-dependent coupling: (a) switch on, $K/\gamma = -0.05$, $t_0 = 0$; (b) switch off, $K/\gamma = 0.05$, $t_0 = \tau$. Curve (c) represents $\langle\hat{\sigma}\rangle_t$ for the time-independent case [$h(t) = 1$]. In all the cases $\tau\gamma = 50$, $n = 40$, $\theta = 0$, the field is initially in a thermal state with $\langle\hat{\Delta}^0\rangle_0 = 0.3$, the two-level system is in the excited state, and we use noninteracting IC.

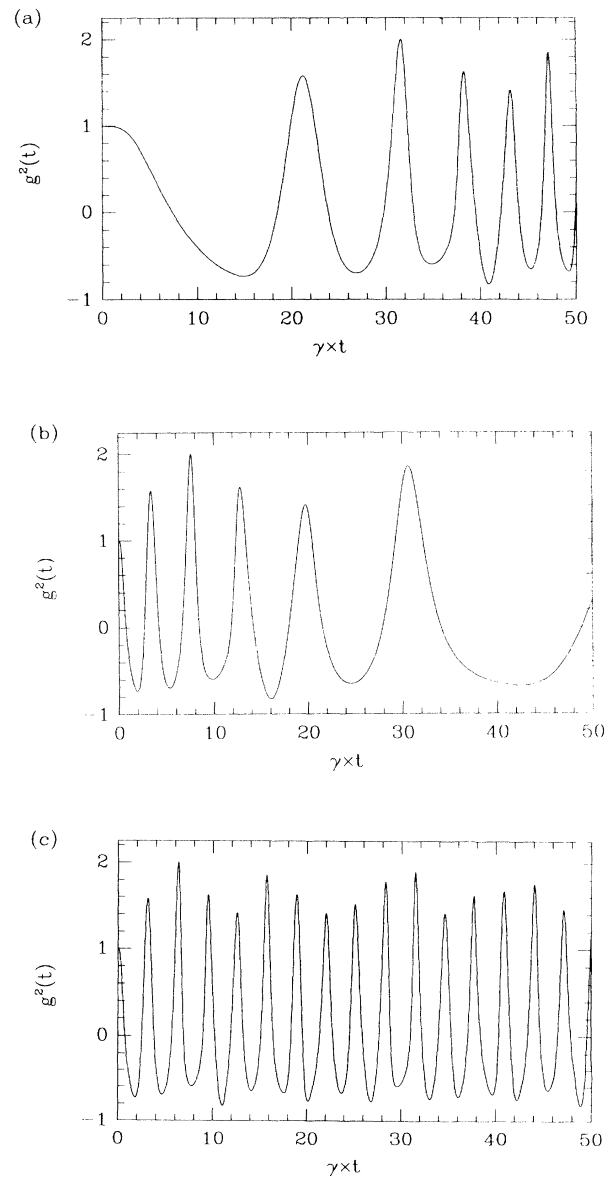

FIG. 9. Temporal evolution of $g^2(t)$ with an exponential time-dependent coupling: (a) switch on, $K/\gamma = -0.05$, $t_0 = 0$; (b) switch off, $K/\gamma = 0.05$, $t_0 = \tau$. Curve (c) represents $g^2(t)$ for the time-independent case [$h(t) = 1$]. In all the cases $\tau\gamma = 50$, $n = 40$, $\theta = 0$, the field is initially in a thermal state with $\langle\hat{\Delta}^0\rangle_0 = 0.3$, the two-level system is in the excited state, and we use noninteracting IC.

case [Fig. 6(c)], as was predicted in Ref. [25]. On the contrary, during the switch off process [Fig. 6(b)] we observe that almost the same behavior is observed during the collapse interval of time. However, the first revival appears later and the period of the oscillations is enlarged. In Fig. 7(a) we show $g^2(t)$. The behavior is similar to the one described in Fig. 6. In Figs. 8 and 9 we plot the evolution of $\langle\hat{\sigma}\rangle_t$ and $g^2(t)$, respectively, for noninteracting IC and the field initially in a thermal state. For this case the IC are $\langle\hat{\sigma}^j\rangle_0 = 0.3^j j!$, $\langle\hat{\Delta}^j\rangle_0 = 0.3^{j+1}(j+1)!$, $\langle\hat{N}^j_{2,1}\rangle_0=\langle\hat{F}^j\rangle_0=\langle\hat{I}^j\rangle_0 = 0$ for $0 \le j \le n$. We took $n = 40$, $\theta = 0$, $\tau\gamma = 50$. In Figs. 8(a) and 8(b) we plot the switch on-off processes, and in Fig. 8(c) the time-independent case. We observe the same behavior obtained for the field in a coherent state.

Thus, for the time-dependent case we observe a general behavior that is independent of the time dependence: as we increase (decrease) the coupling constant by means of a linear or exponential function of time, an increase (decrease) in the frequency of the oscillations is observed. These results, which were observed for $\langle\hat{\sigma}\rangle_t$ and $g^2(t)$, are valid for the other RO. We have also seen that a finite number of equations are needed to obtain exact numerical results since only a finite number of correlations contribute essentially to the solution.

VII. CONCLUSIONS

Summarizing, we have presented a generalized version of the JCH giving a description in terms of physically relevant operators. Since an arbitrary function of time has been included, this formalism allows us to study the system even when the coupling is time dependent. Particularly, the solutions presented in Refs. [25,26] are also included in our formalism, as a particular case of our general results. The advantages of our approach results from the following facts: (a) we have given a description of the system in terms of three sets of relevant operators providing a way to handle the problem of data given in terms of different physical magnitudes; (b) for one of these sets of relevant operators the temporal evolution equations have been shown; (c) we have obtained invariants of motion in terms of the Lagrange multiplier (i.e., intensive variables) which can only be constructed using MEP; (d) the invariants of the motion we have found restrict the possible values of the initial conditions; (e) we have found two sets of dynamical invariants [one for the time-dependent case and the other for $h(t) = 1$], and expressed the nth-order coherence functions in terms of these invariants of the motion (this means that the relationships established for the initial state remain valid for all the temporal evolution of the system, even if the Hamiltonian is considered as a time-dependent one); (f) these initial mean values have been properly evaluated using a MEP density operator (as it was pointed out previously [20,18], the initial conditions play a role as important as the dynamics itself, although not all the formalisms are in position to distinguish clearly which are the pertinent and coherent set of initial conditions); (g) an extension of a sort of quantized version of the Bloch sphere in the dual space of Lagrange multipliers has been obtained, converting the original noncommutative operators structure into geometrical relationships [see Eqs. (36–38)]; (h) the importance of the correlations in the *generalized time-independent* JCH has been shown. Finally, we want to stress the fact that using the MEP approach we have naturally obtained that the nth-order coherence functions of the field (see, for example, Ref. [30], pp. 327–330) are relevant operators. So, we think that this embodying approach can give insight to the Jaynes-Cummings Hamiltonian problem.

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