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Nontrivial dynamics induced by a nonlinear Jaynes–Cummings Hamiltonian

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Abstract

The addition of a nonlinear term to the Jaynes–Cummings Hamiltonian induced a nontrivial discrete dynamics for the number of possible transitions of a given order, represented by a Fibonacci series We describe the physics of the problem in terms of relevant operators which close a semi-Lie algebra under commutation with the Hamiltonian and therefore extending the generalized Bloch equations, already obtained for the linear case, to the nonlinear one. The initial conditions as well as a thermodynamical treatment of the problem is analyzed via the maximum entropy principle density operator Finally, a generalized solution for the time-independent case is obtained and the solution for the field in a thermal state is recovered.

1. Introduction

The combination of two solvable models: a Kerr medium inside a cavity, usually modeled by an anharmonic oscillator [1,2] and the Jaynes–Cummings Hamiltonian (JCH) [3,4] has been recently studied [5–8]. As was shown by Bužek and Jex [5] this combination leads to a solvable model, which describes the interaction of a two-level system with a single mode of an electromagnetic field in the presence of a Kerr-like medium.

Recently, an extension of the Jaynes–Cummings Hamiltonian to the case of time-dependent coupling [9–11] was studied in the frame of the maximum entropy principle (MEP) formalism [12–16]. In that work [9], the dynamical and thermodynamical aspects of this problem were developed in terms of sets of *relevant operators* (RO), obtaining, among other results, a general solution for the temporal evolution of the population inversion. It is also shown that this infinite set of RO governs the dynamics of the system.

In this Letter, we consider the time-dependent generalization of the Jaynes–Cummings Hamiltonian with an additional Kerr-like medium. The main results are (a) The addition of a nonlinear term to the Jaynes–Cummings Hamiltonian induced a nontrivial discrete dynamics for the number of possible transitions of a given order represented by the number of links between relevant operators given by Fibonacci series (b) The dynamical description of the system is done in terms of the mean values of the same physical RO previously obtained for the nonlinear case [9], since the addition of this nonlinearity *does not add* new operators to the semi-Lie algebra (c) A gen-

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eralized version of the quantum Bloch equation for this nonlinear case is *straightforwardly* obtained. (d) A thermodynamical treatment is given via the MEP density operator which allows us to find, properly, the initial conditions of those operators which appears as a consequence of the closure relation of the semi-Lie algebra (e) Finally it is shown how these ideas work out, solving the time-independent case, where the nonlinearity leads to a rich physical interpretation of the dynamical evolution.

In order to put the reader in context we give a brief review of the main concepts of the MEP formalism [12–14] Given the expectation values $\langle\widehat{O}_j\rangle$ of the operators $\widehat{O}_j$, the statistical operator $\hat{\rho}(t)$ [12–14] is defined by

$$\hat{\rho}(t) = \exp\left(-\lambda_0\widehat{I} - \sum_{j=1}^{M}\lambda_j\widehat{O}_j\right), \tag{1}$$

where M is a natural number or infinity, and the $M+1$ Lagrange multipliers λ_j, are determined to fulfill the set of constraints

$$\langle\widehat{O}_j\rangle = \mathrm{Tr}[\hat{\rho}(t)\widehat{O}_j], \quad j = 0, 1, \quad , M \tag{2}$$

($\widehat{O}_0 = \widehat{I}$ is the identity operator) The entropy, defined in units of the Boltzmann constant, is given by

$$S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho}\ln\hat{\rho}] = \lambda_0\widehat{I} + \sum_{j=1}^{M}\lambda_j\langle\widehat{O}_j\rangle, \tag{3}$$

and the time evolution of the statistical operator is given by

$$i\hbar\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = [\widehat{H}(t), \hat{\rho}(t)] \tag{4}$$

One should find the RO entering Eq. (1) so as to guarantee not only that S is maximum, but also a constant of motion. Introducing the natural logarithm of Eq (1) into Eq. (4) it can be easily verified that the RO are those that close a semi-Lie algebra under commutation with the Hamiltonian $\widehat{H}$, i e

$$[\widehat{H}(t), \widehat{O}_i] = i\hbar\sum_{i=0}^{L} g_{ji}(t)\widehat{O}_j \tag{5}$$

Eq (5) defines an $L \times L$ matrix G and constitutes the central requirement to be fulfilled by the operators entering in the density matrix The Liouville equation (4) can be replaced by a set of coupled equations for either the mean values of the RO or the Lagrange multipliers as follows [13],

$$\frac{\mathrm{d}\langle\widehat{O}_j\rangle_t}{\mathrm{d}t} = -\sum_{i=0}^{L} g_{ij}\langle\widehat{O}_i\rangle, \quad j = 0, 1, \quad , L, \tag{6}$$

$$\frac{\mathrm{d}\lambda_j}{\mathrm{d}t} = \sum_{i=0}^{L}\lambda_i g_{ji}, \quad j = 0, 1, \quad , L \tag{7}$$

In the MEP formalism, the mean value of the operators and the Lagrange multipliers belong to dual spaces which are connected by [13]

$$\langle\widehat{O}_j\rangle = -\frac{\partial\lambda_0}{\partial\lambda_j} \tag{8}$$

2. Physically relevant operators, evolution equations, invariants of motion and initial conditions

The generalized time-dependent JCH in the presence of a Kerr-like medium and the adiabatic approximation in the rotating wave approximation takes the form [5]

$$\widehat{H} = E_1\hat{b}_1^\dagger\hat{b}_1 + E_2\hat{b}_2^\dagger\hat{b}_2 + \omega\hat{a}^\dagger\hat{a} + \chi(a^\dagger)^2(\hat{a})^2 + T(t)(\gamma\hat{a}\hat{b}_1\hat{b}_2^\dagger + \gamma^*\hat{b}_2\hat{b}_1^\dagger\hat{a}^\dagger) \tag{9}$$

($\hbar = 1$), where γ is the coupling constant between the system and the external field, E_j and ω are the energies of the levels and the field, respectively, $\hat{a}^\dagger$, $\hat{a}$, are boson operators, $\hat{b}_j^\dagger$ and $\hat{b}_j$ are fermion operators and $T(t)$ is an arbitrary function of time.

As was already obtained for the linear case [9], the operators which fulfill Eq. (5) are constructed using the following physically RO,

$$\widehat{N}_1 = \hat{b}_1^\dagger\hat{b}_1, \tag{10a}$$

$$\widehat{N}_2 = \hat{b}_2^\dagger\hat{b}_2, \tag{10b}$$

$$\widehat{\Delta} = \hat{a}^\dagger\hat{a}, \tag{10c}$$

$$\widehat{I} = \gamma\hat{a}\hat{b}_1\hat{b}_2^\dagger + \gamma^*\hat{b}_2\hat{b}_1^\dagger\hat{a}^\dagger, \tag{10d}$$

$$\widehat{F} = i(\gamma\hat{a}\hat{b}_1\hat{b}_2^\dagger - \gamma^*\hat{b}_2\hat{b}_1^\dagger\hat{a}^\dagger), \tag{10e}$$

$$\widehat{N}_{2,1} = \hat{b}_2^\dagger\hat{b}_2\hat{b}_1^\dagger\hat{b}_1 \tag{10f}$$

$\widehat{N}_l$, $l = 1, 2$, and $\widehat{\Delta}$ can be thought of as the population number of the levels and the external field, respectively. $\hat{I}$ can be considered as the interaction energy between the levels and the external the field, $\hat{F}$ as the particle current between levels and, finally, $\hat{N}_{2,1}$ as the double occupation number. One set which closes a semi-Lie algebra with the Hamiltonian (see Eq. (5)) reads

$$\hat{N}_1^n = (\hat{a}^\dagger)^n \hat{N}_1 (\hat{a})^n, \quad (11a)$$

$$\hat{N}_2^n = (\hat{a}^\dagger)^n \hat{N}_2 (\hat{a})^n, \quad (11b)$$

$$\hat{\Delta}^n = (\hat{a}^\dagger)^n \hat{\Delta} (\hat{a})^n, \quad (11c)$$

$$\hat{I}^n = (\hat{a}^\dagger)^n \hat{I} (\hat{a})^n, \quad (11d)$$

$$\hat{F}^n = (\hat{a}^\dagger)^n \hat{F} (\hat{a})^n, \quad (11e)$$

$$\hat{N}_{2,1}^n = (\hat{a}^\dagger)^n \hat{N}_{2,1} (\hat{a})^n, \quad (11f)$$

$n = 0, 1, \ldots$ The set of RO defined by Eqs (11) has been previously obtained for the linear case [9]. As was said in Ref [9], this set can have the following physical interpretation: We can consider the mean value of the operators with $n > 1$ as a measure of virtual transitions between different states of the field due to the absorption of more than one photon and then emission of the extra photons in a transition between the levels. This interpretation arises from the fact that the powers of $\hat{a}^\dagger$ and $\hat{a}$ represent successive creation and annihilation of photons in the field As the Hamiltonian we are studying now has also a *quadratic term* in field operators, transitions between different states of the field are achieved by more than one path. The number of possible paths or links between RO is given by the Fibonacci series $\mathcal{F}(n)$ [18], where n is the state or level to be achieved, as we will discuss in the followings section In Fig 1 we show all three possible different paths for transitions from levels 0 to 3. Path $\mathcal{A}$ (the JC path) contains only single transitions and therefore three steps are needed to achieve state 3 Paths $\mathcal{B}$ and $\mathcal{C}$ (the nonlinear JC paths) arise from the nonlinear term. From Fig. 1 we see that state 3 is achieved only in two steps (paths $\mathcal{B}$ and $\mathcal{C}$), and therefore earlier. These new paths induced a nontrivial evolution as we will see in Section 3

Although we have already shown [9] that it is possible to find another two sets of RO which satisfy Eq. (5), we restrict ourselves to the one defined by Eqs. (11), which also satisfies Eq. (5) for the general case of having $\chi(\hat{a}^\dagger)^n(\hat{a})^n$ instead of $\chi(\hat{a}^\dagger)^2(\hat{a})^2$ This fact follows exclusively from the commutation relations of the field operators.

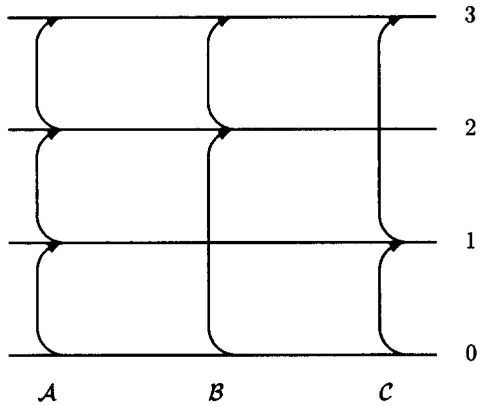

Fig 1 All possible paths for $n = 3$

The dynamical equations for the operators (11) can be obtained using the Ehrenfest theorem (Eq. (6)), and are given by

$$\frac{d\langle \hat{N}_1^n \rangle}{dt} = T(t)\langle \hat{F}^n \rangle + nT(t)\langle \hat{F}^{n-1} \rangle, \quad (12a)$$

$$\frac{d\langle \hat{N}_2^n \rangle}{dt} = -T(t)\langle \hat{F}^n \rangle, \quad (12b)$$

$$\frac{d\langle \hat{F}^n \rangle}{dt} = -(\alpha - 2n\chi)\langle \hat{I}^n \rangle + 2\chi\langle \hat{I}^{n+1} \rangle + 2|\gamma|^2 T(t)[(n+1)\langle \hat{N}_2^n \rangle - \langle \hat{N}_1^{n+1} \rangle + \langle \hat{N}_2^{n+1} \rangle - (n+1)\langle \hat{N}_{2,1}^n \rangle], \quad (12c)$$

$$\frac{d\langle \hat{I}^n \rangle}{dt} = (\alpha - 2n\chi)\langle \hat{F}^n \rangle - 2\chi\langle \hat{F}^{n+1} \rangle, \quad (12d)$$

$$\frac{d\langle \hat{\Delta}^n \rangle}{dt} = (n+1)T(t)\langle \hat{F}^n \rangle, \quad (12e)$$

$$\frac{d\langle \hat{N}_{2,1}^n \rangle}{dt} = 0, \quad (12f)$$

$n = 0, 1, \ldots$, where $\alpha = E_2 - E_1 - \omega$. Eqs. (12) are the *exact* dynamical evolution equations of the RO for the generalized time-dependent JCH with an additional Kerr-like medium. Observe that the dynamical equations include new terms proportional to χ, even though the set of RO has remained the same. Note that for $\chi = 0$ Eqs. (12) reduce to the linear ones [9] Also in this case Eqs. (12) can be thought of as a kind of generalized Bloch equations for the nonlinear case. From Eqs. (12a), (12b), (12e) and (12f) it follows that

$$\{\langle \widehat{N}_1^n\rangle + \langle \widehat{N}_2^n\rangle - \langle \widehat{\Delta}^{n-1}\rangle\}_{n=0}^{\infty}, \quad (13)$$

$$\{\langle \widehat{N}_{2,1}^n\rangle\}_{n=0}^{\infty}, \quad (14)$$

are constants of motion Note that the particle current between levels is equal to the photon flux. For $n = 0$ we obtain again the conservation of the levels population and for $n > 0$ we obtain a restriction for the correlations Into the MEP context, the initial conditions play an important role, and can be obtained in a very well-prescribed procedure, provided the density matrix can be expressed in a diagonal form. In order to derive a thermodynamical or nonzero-temperature approach to the problem at hand, we will write the density matrix including the Hamiltonian as a relevant operator Then, the statistical operator (1) can be written as

$$\hat{\rho}(t) = \exp\Bigg(-\lambda_0\widehat{I} - \beta\widehat{H} - \sum_{n=0}^{\infty}(\lambda_1^n\widehat{N}_1^n + \lambda_2^n\widehat{N}_2^n + \lambda_3^n\widehat{F}^n + \lambda_4^n\widehat{I}^n + \lambda_5^n\widehat{N}_{2,1}^n + \lambda_6^n\widehat{\Delta}^n)\Bigg) \quad (15)$$

Diagonalizing and evaluating the trace of $\hat{\rho}(t)$ we arrive at the following expression for λ_0 in terms of the other Lagrange multipliers,

$$\lambda_0 = \ln\Bigg(\sum_{r=1}^{\infty}\exp(-K_{1,r}) \times 2\cosh(K_{2,r}) + \exp(-\beta E_1 - \lambda_1^0) + \sum_{r=0}^{\infty}\exp(-K_{3,r}) + \sum_{r=0}^{\infty}\exp(-K_{4,r})\Bigg), \quad (16)$$

where

$$K_{1,r} = \tfrac{1}{2}\beta[E_2 + E_1 + (2r-1)\omega + 2\chi(r-1)^2] + \sum_{n=0}^{r}[\tfrac{1}{2}\lambda_1^n\Pi_r^{n-1} + \tfrac{1}{2}\lambda_2^n + \tfrac{1}{2}(2r-n-1)\lambda_6^n\Pi_{r-1}^{n-1}], \quad (17a)$$

$$K_{2,r} = \sqrt{X_r^2 + Y_r^2 + Z_r^2}, \quad (17b)$$

$$K_{3,r} = \beta r\omega + \sum_{n=0}^{r}\lambda_6^n\Pi_r^n, \quad (17c)$$

$$K_{4,r} = \beta(E_2 + E_1 + r\omega) + \chi r(r-1)\sum_{n=0}^{r}\Pi_r^{n-1}[\lambda_1^n + \lambda_2^n + \lambda_5^n + (r-n)\lambda_6^n] \quad (17d)$$

are *invariants of the motion* (this can be shown using Eq. (7)),

$$X_r = \sqrt{r}|\gamma|\Bigg(\beta T(t) + \sum_{n=0}^{r}\lambda_4^n\Pi_{r-1}^{n-1}\Bigg), \quad (18a)$$

$$Y_r = \sqrt{r}|\gamma|\sum_{n=0}^{r}\lambda_3^n\Pi_{r-1}^{n-1}, \quad (18b)$$

$$Z_r = \tfrac{1}{2}[-\beta\alpha - 2\chi(r-1)] + \sum_{n=0}^{r}\{\tfrac{1}{2}\lambda_1^n\Pi_r^{n-1} - \tfrac{1}{2}[\lambda_2^n - (n+1)\lambda_6^n]\Pi_{r-1}^{n-1}\}, \quad (18c)$$

and $\Pi_r^m \equiv \prod_{j=0}^{m}(r-j)$, $\Pi_r^{-1} \equiv 1$.

The initial conditions can be determined using Eqs (8) and (15) For example, the initial mean value of the population of the level 1 and its correlations with the field is

$$\langle \widehat{N}_1^n\rangle_0 = \exp(-\lambda_0)\Bigg\{\exp(-\beta E_1 - \lambda_1^0)\delta_{n,0} + \sum_{r=1}^{\infty}\Pi_r^{n-1}\Bigg[\exp(-K_{1,r})\Bigg(\cosh(K_{2,r}) - \frac{Z_r}{K_{2,r}}\sinh(K_{2,r})\Bigg)\Bigg] + \sum_{r=0}^{\infty}\exp(-K_{4,r})\Bigg\} \quad (19)$$

δ is the Kronecker function Thus, Eq. (15) gives the exact thermodynamical solution for the generalized time-dependent JCH with a Kerr-like medium.

3. Time independent case: nonlinearity and the Fibonacci series

Now, in order to show how this ideas can be applied, we consider the Hamiltonian (9) in the time-independent case. Thus, the Hamiltonian reads

$$\widehat{H} = E_1\hat{b}_1^\dagger\hat{b}_1 + E_2\hat{b}_2^\dagger\hat{b}_2 + \omega\hat{a}^\dagger\hat{a} + \chi(a^\dagger)^2(\hat{a})^2 + \gamma\hat{a}\hat{b}_1\hat{b}_2^\dagger + \gamma^*\hat{b}_2\hat{b}_1^\dagger\hat{a}^\dagger \quad (20)$$

Following Ref. [5] we define the generalized Rabi frequency

$$\delta_n^2 = (\alpha - 2n\chi)^2 + 4(n+1)|\gamma|^2 \tag{21}$$

The evolution of the mean values of the RO can be obtained via the temporal series expansion

$$\langle\widehat{O}\rangle_t = \langle\widehat{O}\rangle_0 + \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{i\hbar}\right)^n \langle\underbrace{[\widehat{O},\widehat{H}],\ldots,\widehat{H}]]}_{n\ \text{times}}\rangle_0, \tag{22}$$

by taking advantage of the semi-Lie structure (Eq. (5)). As an example we evaluate the evolution of $\langle\widehat{N}_1^0\rangle_t$ (Eq. (29)). The linear and quadratic temporal term in Eq. (22) are obtained using Eq. (12a) and Eq. (12c), respectively, with $n = 0$. In evaluating the higher order commutation relations it can be used that

$$[[\widehat{F}^n,\widehat{H}],\widehat{H}] = \delta_n^2\widehat{F}^n + (\delta_{n+1}^2 - \delta_n^2)\widehat{F}^{n+1} + 4\chi^2\widehat{F}^{n+2}. \tag{23}$$

Thus, after cumbersome algebra (which we will publish elsewhere) Eq. (22) can be rewritten as an infinite series, whose terms are proportional to mean values of the RO $\langle\widehat{O}^n\rangle$ ($\widehat{O}^n$ given by Eqs. (11)) multiplied by a functional $\mathcal{L}^n(\bar{y})$ (Eq. (28)) which depends on all the frequencies associated with the RO ($\bar{y}$ is a vector with components $[\cos(\delta_k) - 1]/\delta_k^2$ or $\sin(\delta_k)/\delta_k$, depending on the particular RO $\widehat{O}$, as will be explained in the following paragraphs)

In the JCH, $\chi = 0$, and $\mathcal{L}^n(\bar{y})$ reduces to a sum over terms depending on all the frequencies δ_0, δ_1, , δ_n (see Eq. (24) and path $\mathcal{A}$ in Fig. 1) because the Hamiltonian is linear. As the Hamiltonian we are studying now has a linear and a quadratic term in field operators, the links between RO can only consist of those corresponding to one or two level jumps (see Eq. (23)). Before going into the general form of the functional we give its explicit form for $n = 3$. Thus $\mathcal{L}^3(\bar{y})$ reads

$$\begin{aligned}\mathcal{L}^3(\bar{y}) = {} & \underbrace{(\delta_3^2 - \delta_2^2)(\delta_2^2 - \delta_1^2)(\delta_1^2 - \delta_0^2)}_{\text{path } \mathcal{A}} \\ & \times \sum_{k=0}^{3} \tilde{a}_{3,k}(\delta_0,\delta_1,\delta_2,\delta_3)\frac{\cos(\delta_k t) - 1}{\delta_k^2} \\ & + \underbrace{4\chi^2(\delta_2^2 - \delta_3^2)}_{\text{path } \mathcal{B}} \sum_{k=0}^{2} \tilde{a}_{2,k}(\delta_0,\delta_2,\delta_3)\frac{\cos(\delta_{\imath_k} t) - 1}{\delta_{\imath_k}^2} \\ & + \underbrace{4\chi^2(\delta_1^2 - \delta_0^2)}_{\text{path } \mathcal{C}} \sum_{k=0}^{2} \tilde{a}_{2,k}(\delta_0,\delta_1,\delta_3)\frac{\cos(\delta_{\imath_k} t) - 1}{\delta_{\imath_k}^2},\end{aligned} \tag{24}$$

where $\imath_k$ is the kth component of $\tilde{a}$, and

$$\tilde{a}_{3,k}(\delta_0,\delta_1,\delta_2,\delta_3) = (-1)^{3+k+1}\frac{|\mathcal{V}_{3,k}|}{|\mathcal{V}|}, \tag{25}$$

$\mathcal{V}_{n,k}$ being the reduced n,k matrix of the Vandermonde matrix $\mathcal{V}$ [17],

$$\mathcal{V} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \delta_0^2 & \delta_1^2 & \delta_2^2 & \delta_3^2 \\ \delta_0^4 & \delta_1^4 & \delta_2^4 & \delta_3^4 \\ \delta_0^6 & \delta_1^6 & \delta_2^6 & \delta_3^6 \end{pmatrix} \tag{26}$$

It is important to notice that the weight (paths $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in Eq. (24)) multiplying each sum in Eq. (24) depends on the path. This is not the case in the JCH because when $\chi = 0$ $(\delta_{n+1}^2 - \delta_n^2)$ (and the determinant of $\mathcal{V}$) is a n-independent constant In the general case the number of different paths (or links) $\mathcal{F}(n)$ of order n are obtained from the Fibonacci series. Each path can be represented by a $(n+1)$-upla, $\bar{x}_{n,l}$, of zeros and ones, where a 1 (0) appears in each frequency that is (not) included in this path Thus, for example, we obtain

$$\begin{aligned} & n = 0: \ \bar{x}_{0,1} = 1, \\ & n = 1 \ \ \bar{x}_{1,1} = 1,1, \\ & n = 2\cdot \ \bar{x}_{2,1} = 1,1,1,\ \bar{x}_{2,2} = 1,0,1, \\ & n = 3 \ \ \bar{x}_{3,1} = 1,1,1,1,\ \bar{x}_{3,2} = 1,0,1,1, \\ & \qquad \bar{x}_{3,3} = 1,1,0,1, \end{aligned} \tag{27}$$

etc So, all the different paths, or links of order n are $\{\bar{x}_n\} = \{\{\bar{x}_{n-1}\},1\} \cup \{\{\bar{x}_{n-2}\},0,1\}$. Now, the general form of $\mathcal{L}^n(\bar{y})$ reads

$$\mathcal{L}^n(\bar{y}) = \sum_{l=1}^{\mathcal{F}(n)} \left(\prod_{j=1}^{n} \phi_{n,l,j}\right)\left(\prod_{k=1}^{n} \psi_{n,l,k}\right) \times \sum_{r=0}^{s_{n,l}-1} \tilde{a}_{s_{n,l}-1,r}(\delta_{i_0}, \ldots, \delta_{i_{s_{n,l}-1}}) y_{i_r}, \tag{28}$$

where $s_{n,l} = \sum_{k=0}^{n} x_{n,l,k}$ gives the number of frequencies associated with a given path, n determines the correlation order, l the path, k the frequency,

$$\begin{aligned} \phi_{n,l,j} &= (\delta_j^2 - \delta_{j-1}^2) \quad \text{if } x_{n,l,j} = 1,\ x_{n,l,j-1} = 1, \\ &= 1 \quad \text{otherwise}, \end{aligned}$$

$$\begin{aligned} \psi_{n,l,k} &= 4\chi^2 \quad \text{if } x_{n,l,k-1} = 1,\ x_{n,l,k} = 0, \\ &= 1 \quad \text{otherwise}, \end{aligned}$$

and $y_{i_r} = [\cos(\delta_{i_r} t) - 1]/\delta_{i_r}^2$ ($y_{i_r} = \sin(\delta_{i_r} t)/\delta_{i_r}$) are the components of $\bar{y} \equiv C/\delta^2$ ($\bar{y} \equiv S/\delta$) So, the exact solution for $\langle \widehat{N}_1^0 \rangle_t$ reads

$$\begin{aligned} \langle \widehat{N}_1^0 \rangle_t &= \langle \widehat{N}_1^0 \rangle_0 \\ &- \sum_{n=0}^{\infty} [\langle \widehat{F}^n \rangle_0 \mathcal{L}^n(S/\delta) + \langle \widehat{A}^n \rangle_0 \mathcal{L}^n(C/\delta^2)] \\ &+ \sum_{n=1}^{\infty} [2(\chi \langle \widehat{I}^n \rangle_0 + |\gamma|^2 \langle \widehat{N}_2^n \rangle_0) \mathcal{L}^{n-1}(C/\delta^2)], \end{aligned} \tag{29}$$

where $\langle \widehat{A}^n \rangle_0 = (\alpha - 2n\chi)\langle \widehat{I}^n \rangle_0 + 2|\gamma|^2[\langle \widehat{N}_1^{n+1} \rangle_0 - (n+1)(\langle \widehat{N}_2^n \rangle_0 - \langle \widehat{N}_{2,1}^n \rangle_0)]$ The same results can be obtained for all the RO.

As can be seen, the nonlinear nature of this problem brings nontrivial behavior since the number of different sums multiplying the RO of order n in the exact solution, corresponding to all the possible paths between levels 0 and n is given by $\mathcal{F}(n)$ Eq (29) can be simplified if one notices the particular Fibonacci-like properties of the frequencies δ_n,

$$\delta_{n+1}^2 = 2\delta_n^2 - \delta_{n-1}^2 + 8\chi^2, \tag{30}$$

or

$$\delta_k^2 - \delta_j^2 = (k-j)(\delta_1^2 - \delta_2^2) + [k(k-1) - j(j-1)]4\chi^2 \tag{31}$$

We obtain

$$\begin{aligned} \langle \widehat{N}_1^0 \rangle_t &= \langle \widehat{N}_1^0 \rangle_0 \\ &- \sum_{n=0}^{\infty} \left(\langle \widehat{F}^n \rangle_0 \sum_{k=0}^{n} a_{n,k} \frac{S_k}{\delta_k} + \langle \widehat{A}^n \rangle_0 \sum_{k=0}^{n} a_{n,k} \frac{C_k}{\delta_k^2} \right) \\ &+ \sum_{n=1}^{\infty} \left(2(\chi \langle \widehat{I}^n \rangle_0 + |\gamma|^2 \langle \widehat{N}_2^n \rangle_0) \sum_{k=0}^{n-1} a_{n-1,k} \frac{C_k}{\delta_k^2} \right), \end{aligned} \tag{32}$$

where $a_{n,k} = (-1)^{n+k+1}/[(n-k)!k!]$, as in the case of the JCH (see Ref. [9]). Eq. (32) has the same form as the one given in Ref [9] for the JCH if we replace δ_k by Ω_k, the Rabi frequency [19] and neglect $2\chi\langle \widehat{I}^n \rangle_0$ Even though, the physics of the JCH with a Kerr medium is completely different due to the presence of the nonlinear term. As was mentioned in Ref [9], for the JCH the temporal function multiplying the correlation $\langle \widehat{O}^n \rangle$ ($\sum_{k=0}^{n} a_{n,k} y_k$) is proportional to t^{2n+1} (i e the first $2n$ terms of the Taylor expansion vanish) *This is not the case when the Kerr medium is present due to the fact that shorter paths exist* (i e. the path with quadratic transitions will link the states 0 and n in $\frac{1}{2}n$ steps).

For the particular case of having zero particles in level 1, one particle in level 2, an electromagnetic field in a thermal state with mean number of photons $\langle \widehat{\Delta} \rangle_0 \neq 0$ (i e. $\lambda_6^0 \neq 0$), we obtain

$$\langle \widehat{N}_1^n \rangle_t = -2|\gamma|^2 \sum_{n=0}^{\infty} \frac{(n+1)C_n}{\delta_n^2} \frac{\langle \widehat{\Delta} \rangle_0^n}{(1 + \langle \widehat{\Delta} \rangle_0)^{n+1}}, \tag{33}$$

which is the result usually shown in the literature [5].

Summarizing, we have studied a generalized time-dependent version of the JCH with an additional Kerr-like medium and we have: (a) given a description of the problem in terms of physically relevant operators, (b) obtained the invariants of motion of this problem, (c) properly evaluated the initial mean values using a MEP density operator, (d) obtained a sort of quantized version of the Bloch sphere in the dual space of Lagrange multipliers, (e) solved exactly the problem for the time-independent case, obtaining new physical insights about the character of the nonlinear effect. We have shown that the nonlinearity of the problem is characterized by a weighted sum over correlations, were the weights are related to the Fibonacci series. We have also shown the way in which the usual results are obtained for an initial thermal state. Finally, we stress that this *new insight* cannot be seen when using the wave-function approach.

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