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Order in the turbulent phase of globally coupled maps

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The broad peaks seen in the power spectra of the mean field in a globally coupled map system indicate a subtle coherence between the elements, even in the "turbulent" phase. These peaks are investigated in detail with respect to the number of elements coupled, nonlinearity and global coupling strengths. We find that this roughly periodic behavior also appears in the probability distribution of the mapping, which is therefore not invariant. We also find that these peaks are determined by two distinct components: effective renormalization of the nonlinearity parameter in the local mappings, and the strength of the mean field interaction term. Finally, we demonstrate the influence of background noise on the peaks, which is quite counterintuitive, as they become *sharper* with increase in strength of the noise, up to a certain critical noise strength.

1. Introduction

Globally coupling in dynamical systems yields a host of very novel features. This class of complex systems is of considerable importance in modeling phenomena as diverse as Josephson junction arrays, vortex dynamics in fluids, and even evolutionary dynamics, biological information processing and neurodynamics. The ubiquity of globally coupled phenomena has thus made it a focus of much recent research activity [1–4].

In this paper we study the globally coupled map (GCM) introduced by Kaneko [2]. It is a dynamical system of N elements consisting of local mappings as well as an additive average-type interaction term, through which the global information influences the individual elements. It is thus analogous to a mean field version of coupled map lattices. The explicit form of the GCM we use is

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f(x_n(j))\,, \qquad (1)$$

where n is a discrete time step and i is the index of the elements ($i = 1, 2, \ldots, N$). The function $f(x)$ was chosen to be the well known dissipative chaotic logistic map

$$f(x) = 1 - ax^2\,. \qquad (2)$$

This choice helps us to make contact with previous results.

The above GCM model has two conflicting trends: destruction of coherence due to the chaotic dynamics of the individual elements, and a kind of synchronization through the global av-

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eraging. For large global coupling, this synchronization may be complete (all elements moving coherently), and appears even in the fully chaotic (in time) regime. On the other hand, for a nonlinearity parameter a such that the local dynamics is strongly chaotic, and in the presence of a small coupling, the behavior of the system is "turbulent". This means that all elements of the lattice behave chaotically in time (all Lyapunov exponents are positive [3]), and that there is no clustering (partial "entrainment" or synchronization). In fact, the distance between any two different elements of the lattice that have at some moment close values grows exponentially with time. (For $\epsilon = 0.1$ and $a = 1.99$ this exponent is ≈ 0.4). So, in practice, the elements of the lattice seem to behave like independent quasirandom variables.

This has led to the following "simplicity" hypothesis: if in this turbulent regime the different elements of the lattice behave in fact as independent quasirandom numbers, then in the $N \to \infty$ limit the mean field h_n, defined by

$$h_n \equiv \frac{1}{N} \sum_{j=1}^{N} f(x_n(j)) , \tag{3}$$

should converge to a fixed value, uncoupling the system. In fact, a similar idea has been used by Kuramoto and others [4] in order to analyze the $N \to \infty$ limit of a globally coupled system of limit cycle oscillators, and its coherent–incoherent transition. This "simplicity" hypothesis can also be cast in the following terms: consider for a moment a system similar to eq. (1), with some fixed a and ϵ, but where we substitute the time dependent mean field h_n by some constant h_{in}. This gives us a lattice of uncoupled logistic maps, which in the $N \to \infty$ has an invariant probability distribution. From this distribution we can evaluate h_{out}, defined as the average value of $f(x)$. Then two questions come immediately to mind: is there a solution for the self-consistency equation $h_{\text{out}}(h_{\text{in}}; a, \epsilon) = h_{\text{in}}$? And if so, is that solution stable under small fluctuations of h_{in}, in the fully coupled model?

Coming back to a finite lattice, we would expect the fluctuations that appear in the system to behave statistically, if this limiting value for h does exist. In particular, we should expect a decay in the mean square deviation (MSD) of the mean field ($\equiv \langle h^2 \rangle - \langle h \rangle^2$) as $1/N$ (law of large numbers), and its distribution to be Gaussian (central limit theorem). These two questions were explored by Kaneko [3] and the results found were that the system in eq. (1) *violated the law of large numbers but not the central limit theorem* (this last affirmation has been reevaluated in ref. [5], where it was found that the tails of the distribution diverged from those of a Gaussian). Even in the fully "turbulent" phase, where there is absolutely no synchronization among the elements, a subtle coherence emerged. This was reflected in the saturation of the MSD, that stopped decaying after some critical lattice size N_c, in the broad peaks that appear in the power spectrum of the mean field h_n, and in the fact that the mutual information of the system remained non-zero for all lattice sizes.

It should be noted that these results are not universal, since there are related systems that show proper statistical behavior. In particular, it has been found [5] that a globally coupled lattice of tent maps (eq. (1) with $f(x) = 1 - a|x|$), behaves as expected in its turbulent regime. The MSD of the mean field dies away as $1/N$ and the Fourier transform of h_n does not develop any peaks.

In section 2 of this paper we examine, through numerical experiments, the transition between the power spectrum of a single $x_n(i)$ (which is only mildly humped) to the spectrum of the collective quantity h_n which displays broad peaks, indicating collective "beats" in its dynamics. Then, in section 3 we examine the behavior of another global quantity, namely the probability distribution of the mapping, for possible similar behaviour. Here too we find evidence of non-statistical behavior, with the emergence of a kind of collective "beating", and a saturation in the fluctuations of the probability values. This clearly means that the probability distribution of the mapping is not invariant. In section 4 we attempt an analysis of this emergence of order in terms of two distinct effects, one due to an effective renormalization of the local nonlinear parameter of the map, and another due to the synchronization induced by the mean field acting over the individual elements. Finally we investigate the influence of noise in the system. The surprising result here is that the peaks in the power spectrum of the mean field get *sharper* as the strength of noise increases, up to a certain critical noise strength. This counterintuitive phenomena are demonstrated through numerical experiments in section 5.

2. Emergence of peaks in the power spectrum

In this section we want to trace the development of the peaks in the power spectrum of the mean field. Clearly, when $N = 1$, i.e., when there is a single logistic map, we have a very flat (aperiodic) spectrum. But even as we put in another element ($N = 2$) we find a "ghost" of the peaks making its presence felt. So, the appearance of one broad peak in the spectrum is almost immediate, as is evident from the power spectra for very low lattice sizes in fig. 1. It takes larger lattices to resolve this peak into its various components. For these spectra we have evaluated the autocorrelation function, which is defined by

$$C = \frac{1}{M} \sum_{i=1}^{M} \frac{\Sigma_{j=1}^{M} P(j + i \bmod M) P(J)}{\Sigma_{j=1}^{M} P(j)\, P(j)}, \tag{4}$$

where $P(j)$ is the value of the power at the jth frequency index, and M is the number of discrete points in the spectrum. This provides a good measure of the "flatness" of a spectrum, with C taking the value 1 when the spectrum is completely flat, and 0 when there are just δ-peaks. A better indicator of the sharpness of the peaks is given by

$$S = -\log_{10} C, \tag{5}$$

where $S = 0$ is the signature of a completely flat spectrum and $S \to \infty$ is the signature of (very sharp) δ-peaks. We find that S increases very fast with increasing lattice size N (size fig. 2), indicating that the peaks emerge rapidly, on

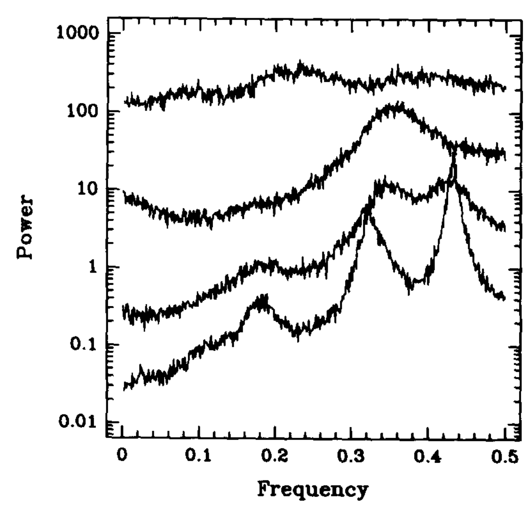

Fig. 1. Power spectra of the mean field for lattice sizes $N = 1, 8, 64$ and 512 (from top to bottom). Here $a = 1.99$, $\epsilon = 0.1$ and we average over 100 runs of 1024 iterations each.

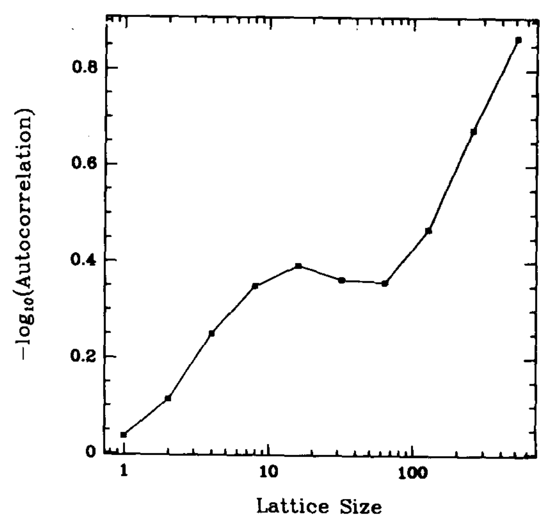

Fig. 2. Measure of the sharpness of the peaks in the power spectra, as defined in the text, vs. lattice size N ($a = 1.99$, $\epsilon = 0.1$).

addition of elements, from the flat spectrum corresponding to a system with a single element.

We also investigate the power spectra of partial sums, given as

$$S_m(n) = \frac{1}{m} \sum_{i=1}^{m} f(x_n(i)) , \tag{6}$$

where the $x_n(i)$ evolve under the effect of the full mean field h_n, as given by eq. (1). The power spectrum for a single element under the influence of the full mean field ($S_1(n)$) shows some influence of the roughly periodic behavior of h_n (see fig. 3). It contains, in any case, much more periodic modulation that the single isolated logistic map, as can be seen by comparing to the topmost spectrum in fig. 1. It is interesting to notice that this behavior remains unchanged for small partial sums, so much so that the different spectra look like parallel displacements of each other, except for the intrusions of the two main frequencies. This suggests that under the influence of the full mean field the partial sums behave as h_n plus some amount of white noise, where the intensity of this noise decays initially as $1/N$. This behavior is quite different from that of the mean field for small lattices, shown in fig. 1.

3. Probability distributions

We now investigate the dynamics of the probability distributions, defined as

$$P_\delta(y;n) = \frac{1}{2\delta N} \sum_{i=1}^{N} \Theta[\delta - |x_n(i) - y|] , \tag{7}$$

for small δ and large N. For a logistic map in the chaotic regime this quantity is invariant in the $N \to \infty$ limit. Here the individual local maps are well inside the chaotic regime (nonlinearity parameter $a = 1.99$). However, the "beating" behavior observed in the mean field should be reflected in the dynamics of the probability distributions, and there is a good possibility that the finite lattice fluctuations in this distribution will not die out with growing N. What numerical experiments show is that indeed this happens, as can be seen in fig. 4, where the MSD of $P(y)$ as

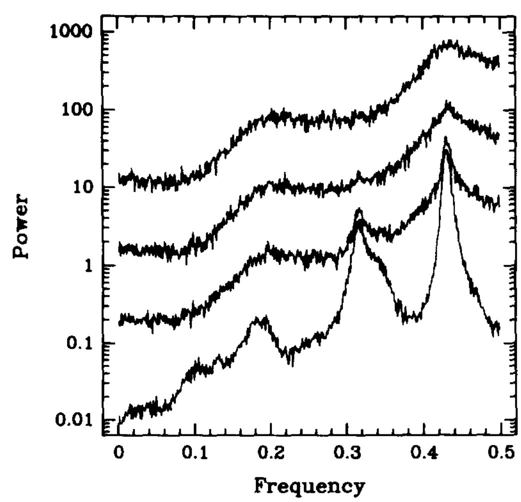

Fig. 3. Power spectra of partial sums S_m, as defined in the text, for $m = 1, 8, 64$ and the full lattice (from top to bottom). Here $a = 1.99$, $\epsilon = 0.1$, $N = 10\,000$ and we average over 100 runs of 1024 iterations each.

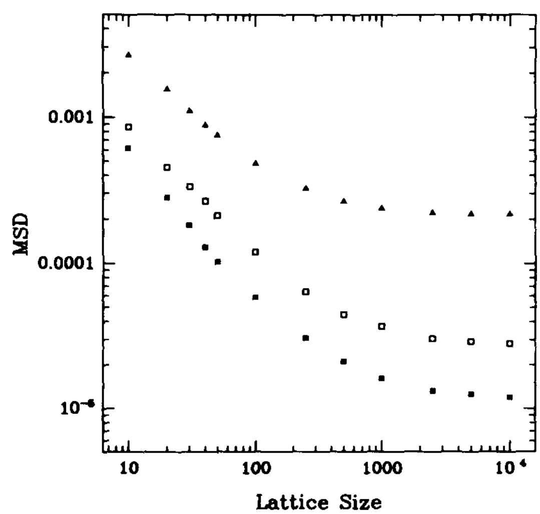

Fig. 4. Mean square deviation of $P_\delta(y;n)$, ($\delta = 0.01$), vs. lattice size N, at three different values of y; $y = 0.0$ (■), $y = 0.5$ (□), and $y = 0.9$ (▲) ($a = 1.99$, $\epsilon = 0.1$, $N = 10\,000$ and the number of iterations is $10\,000$).

a function of N is plotted for three different values of y. (It should be noted that we are considering well populated bins here). It is clear from the plot that, after a critical N, the MSD does not fall as $1/N$ but saturates instead. Therefore, this distribution *does not converge to an invariant distribution* as N grows.

Further, we have noticed that the power spectrum of $P(y;n)$ shows the same broad peaks as the mean field h_n. This can be seen by taking the first few moments of the distribution and doing a spectral analysis. We have done this for the first four moments, and the resulting spectra are almost identical to that of the mean field. On the other hand, we can also follow the time evolution of the probability at a given value of y. Fig. 5 shows the power spectrum of a representative bin, where the peaks are clearly discernible. (There are, however, other bins where the peaks are less pronounced or almost non-existing). Although these spectral curves are not equal to that of the mean field, the peaks on the "beating" bins match with those of h_n, which is not surprising, since these are just different manifestations of the same underlying collective effect.

4. Dependence on the global coupling parameter

It is instructive to study the functional dependence of the MSD on the global coupling parameter ϵ, since it gives the strength of the global averaging, and is in this sense the source of the synchronization effect. Thus, we have checked the value of the MSD of the mean field as a function of ϵ. At first viewing this functional dependence seems very erratic. (see fig. 6a). Moreover, in the explored range of ϵ (0.0–0.2), the maximum value of the MSD was found to be one order of magnitude larger than the value at $\epsilon = 0.1$, where most of the work has been concentrated up to now [3].

We now attempt an explanation of this non-systematic behavior, and in particular of the surprisingly large values of the MSD found in certain small ranges of ϵ. This can partially be accounted for if we consider the effects of the coupling as divided roughly in two components. One is the renormalization of the nonlinear parameter a by the introduction of the multiplicative $1-\epsilon$ term in the individual maps. The other

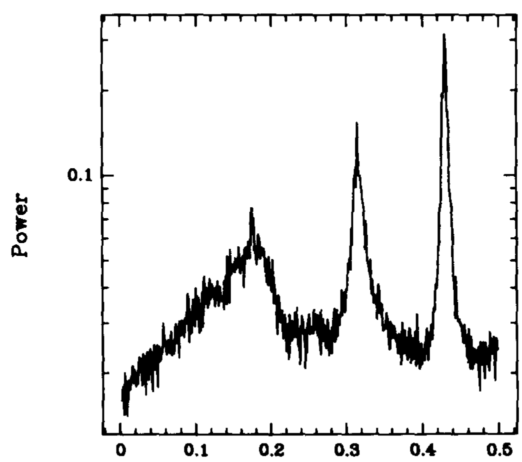

Fig. 5. Power spectrum of $P_\delta(y;n)$ ($\delta = 0.01$) at $y = 0.9$. Here we average over 100 runs of 1024 iterations each ($a = 1.99$, $\epsilon = 0.1$ and $N = 10\,000$).

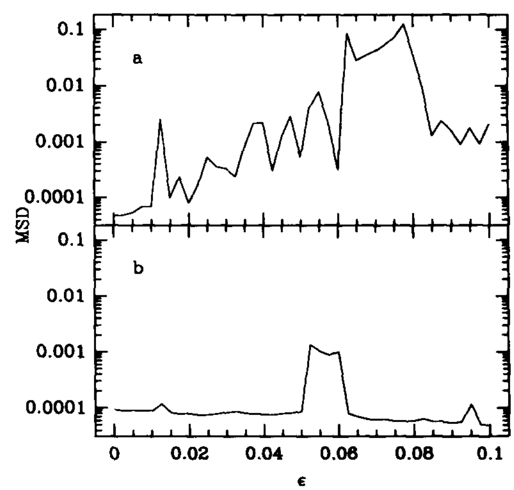

Fig. 6. Mean square deviation vs. global coupling parameter ϵ for (a) the full map, as given in eq. (1) in the text, and (b) a set of uncoupled logistic maps with $a_{\text{eff}} = a(1-\epsilon)^2$ ($a = 1.99$, $N = 10\,000$).

is the action of the mean field, whose effective strength in the dynamics of the individual elements is determined by ϵ. (Notice, however, that the nonlinear parameter a used to construct the mean field remains unaffected by the global coupling ϵ). To check this hypothesis we have explored, as a function of ϵ, the behavior of a set of uncoupled logistic maps with the local nonlinear parameter set to the renormalized value, which is given by

$$a_{\mathrm{eff}} = a(1-\epsilon)^2 . \qquad (8)$$

We have computed the MSD for such a system, and find that its profile is similar to that of the fully coupled maps (see fig. 6b). What is striking here is the appearance of a plateau of large values for the MSD close to a similar plateau in the fully coupled problem. This plateau occurs around $a_{\mathrm{eff}} \approx 1.75$ and corresponds to the 3-window of the logistic map [6]. The width of the plateau is related to the width of the periodic window. Furthermore, a second smaller and narrower sharp peak appears at $a_{\mathrm{eff}} \approx 1.94$, which corresponds to a very narrow 4-window. This shows that there is an influence of the periodic windows of the logistic map in the value of the MSD for the fully coupled problem, through the ϵ-dependent renormalization of the nonlinearity parameter in the local mappings. This hypothesis is further sustained by the fact that the power spectrum of h_n in the fully coupled map, for values of ϵ corresponding to the largest plateau, shows a clearly dominant $\frac{1}{3}$ frequency (see fig. 7), and the power spectrum for ϵ corresponding to the smaller peak shows a clear $\frac{1}{4}$ frequency (see fig. 8).

So the skeleton of the functional dependence of the MSD on coupling comes from the effects of renormalizing the nonlinear parameter in the local maps, which may push them into periodic windows, leading to some synchronization. This synchronization is not complete, because the mean field is still being evaluated at the bare value of a, where the dynamics is strongly chaotic. It is, however, strong enough to produce the narrow ranges of ϵ where the deviation is an order of magnitude larger than elsewhere. But this is clearly not a full explanation of the almost periodic fluctuations of h_n. The MSD for the uncoupled case is much too small compared to that of the fully coupled case, and accounts for only the gross features of the MSD vs. ϵ curve.

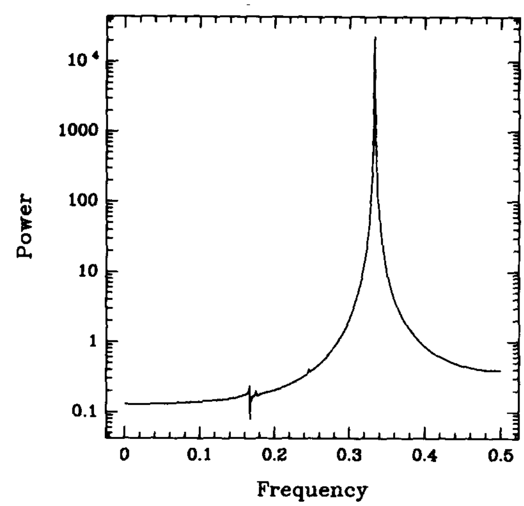

Fig. 7. Power spectrum of h_n at $\epsilon = 0.075$ ($a = 1.99$, $N = 10\,000$). Here we average over 100 runs of 1024 iterations each.

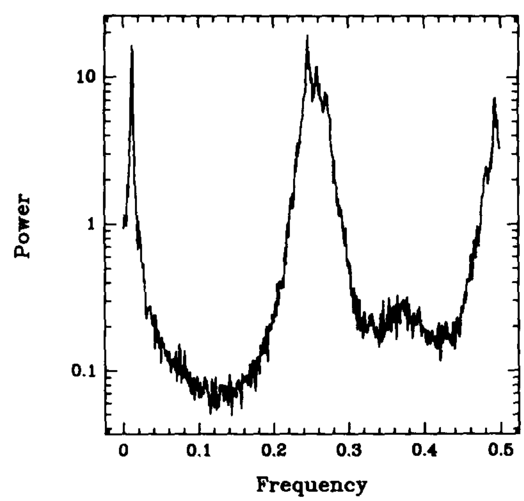

Fig. 8. Power spectrum of h_n at $\epsilon = 0.0125$ ($a = 1.99$, $N = 10\,000$). Here we average over 100 runs of 1024 iterations each.

So, the "flesh" of the MSD comes from the effects of the mean field which lead to synchronization by global averaging. For a full characterization of the broad collective motion of the system one must then take into account both effects. As an extra verification, have also computed the MSD for a system analogous to eq. (1), but where the local maps are not multiplied by the $1-\epsilon$ term, and so there is no renormalization of the nonlinearity parameter. For such a system the effects come solely from the interaction with the mean field, and we find that the MSD, as expected, increases monotonically with ϵ. We have investigated also a realistic physical system that displays the same kind of phenomena, with similar results [7].

5. Effects of noise

We now examine the effects of additive noise in the dynamics of the mean field. For this simulate the system

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f(x_n(j)) + \sigma\eta_n^i , \qquad (9)$$

where η_n^i is a random number uniformly distributed in the interval $[-0.5, 0.5]$. As described in ref. [3], adding noise to the system impedes the saturation of the MSD, but does not restore the normal $1/N$ behavior. Instead, the fluctuations now decay as $1/N^{\alpha}$, with $\alpha < 1$ for σ not too large. This effect appears only for noise strength above some threshold σ_c. We have found that this anomalous decay of the MSD does not mean that the mean field h_n stops being almost periodic. On the contrary, it is found that for values of the added noise up to a value roughly equal to σ_c the sharpness of the power spectrum increases. This counterintuitive behavior can be clearly seen in figs. 9a, b and c, where we have plotted the power spectra for three values of σ, and in fig. 10, which shows the value of S, the measure of sharpness defined in eqs. (4) and (5), vs. σ. Clearly the sharpness increases with increasing noise, up to $\sigma = 0.009$, and decreases from there on. We do not have an explanation of this very surprising phenomenon as yet.

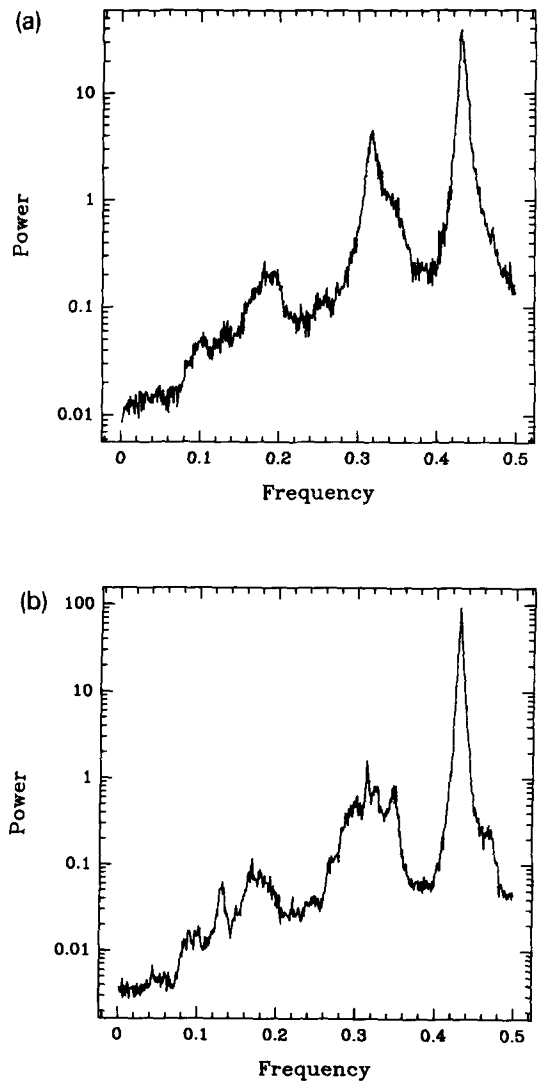

Fig. 9. Power spectra of the mean field in the presence of noise of strength σ = (a) 0.0, (b) 0.004, (c) 0.009 ($a = 1.99$, $\epsilon = 0.1$, $N = 10000$). Here we average over 100 runs of 1024 iterations each.

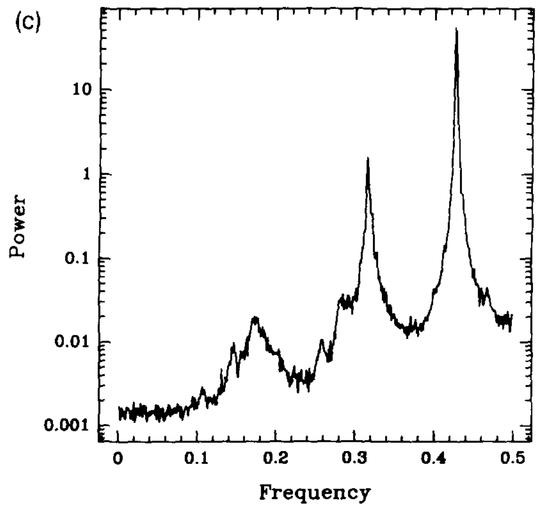

Fig. 9. (cont.).

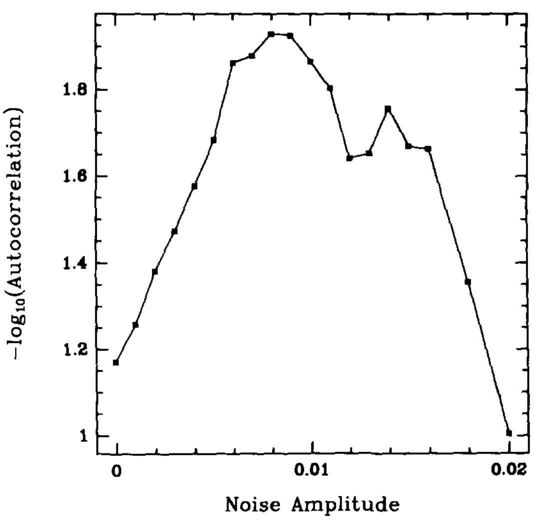

Fig. 10. Measure of the sharpness of peaks in the power spectra, as defined in the text, vs. noise strength σ ($a = 1.99$, $\epsilon = 0.1$, $N = 10\,000$).

6. Conclusions and comments

Here we have investigated various aspects of the dynamics of the mean field in a globally coupled chaotic system. The mean field shows evidence of a rough periodicity as is suggested through the broad, significant peaks in its power spectrum. We trace the development of these peaks with respect to the number of elements coupled, and study their presence in partial averages. Further, we examine another important global quantity, namely the probability distribution, and find that it is *not invariant*, and like the mean field, does not obey the law of large numbers. Moreover, there is evidence of a similar "beating" pattern in its power spectra, with frequencies of this roughly-periodic behavior matching those of the mean field.

Next we find the functional dependence of the mean square deviation of the mean field on the global coupling parameter. We then attempt to decompose the effect we observe as coming from two distinct sources: one, the renormalization of the nonlinearity parameter in the local maps, and second, the contribution from the mean field, which introduces a degree of synchronization. This way of looking at the system helps us account for the extremely large deviations found in certain ranges of the coupling parameter. We can in fact identify the largest plateau in the MSD vs. ϵ graph with the period-3 window, into which the local maps are pushed due to the effective renormalization of a.

Lastly, we explore the effects of noise on the rough periodicities observed in the mean field. We find that the periodicities do in fact persist up to a reasonably large strength of noise. Furthermore, the peaks actually get sharper with increase of the noise strength, up to a critical value. This strange effect is another instance of stabilization of periodic motion through small noise, resembling in a way the phenomenon of stochastic resonance [8]. However, it is not evident that there exists any direct connection between our observations and this other problem.

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