PHYSICAL REVIEW B VOLUME 3, NUMBER 9 1 MAY 1971

Attenuation of High-Frequency Sound in Clean Type-II Superconductors near H_{c2}. I*

Hilda A. Cerdeira and A. Houghton
Department of Physics, Brown University, Providence, Rhode Island 02912
(Received 2 November 1970)

In this paper we calculate the attenuation of high-frequency sound in clean type-II superconductors near the upper critical field H_{c2}. The approach used is a generalization of that used by Brandt, Pesch, and Tewordt to calculate the density of states of a pure type-II superconductor. It is shown that the attenuation is strongly dependent on mean free path even if $l \gg \xi_0$. In the low-temperature limit we obtain a simple functional relationship between attenuation, mean free path, and magnetic field.

I. INTRODUCTION

Recently there has been considerable experimental and theoretical interest in the transport properties of clean ($l \gg \xi_0$, here $l = v_F \tau$ is the electronic mean free path and ξ_0 is the pure superconductor coherence distance) type-II superconductors near the upper critical field H_{c2}. In contrast to the dirty limit $l \ll \xi_0$, where the transport properties near H_{c2} can be well understood in terms of gapless superconductivity,[1] theoretical results in the clean limit have proved much more difficult to obtain. This was first pointed out by Cyrot and Maki[2] who showed that the series expansion in powers of the order paramater, $\Delta(\vec{r})$, usually employed in the dirty limit does not converge in the clean case and leads to unphysical results.

Since then Maki, studying the different terms in the expansion in powers of the order parameter, noticed a similarity with a BCS superconductor carrying a uniform current and made an ansatz[3] based on these similarities. He claimed to sum in this way the most divergent terms of the expansion in powers of $\Delta(\vec{r})$. Maki used this technique to calculate the transport properties of clean type-II superconductors and found, for example, that both the change in ultrasonic attenuation[3] and thermal conductivity[4] at H_{c2} were proportional to $(H_{c2} - B)^{1/2}$ in contrast to the dirty limit[5] which gives a linear dependence. It should be pointed out that the square-root dependence found in the Maki theory occurs because of the similarity with the BCS theory, which yields a linear dependence on Δ. In the meantime, the ultrasonic attenuation has been measured by several groups[6–8] and the thermal conductivity by Vinen *et al.*[8] These experiments appear to be consistent with a square-root dependence near H_{c2}; however, the experimental results are strongly dependent on mean free path which is not predicted by the Maki theory.

The second theoretical approach to the theory of clean type-II superconductors is due to Brandt, Pesch, and Tewordt[9] (BPT). These authors did not use a power-series expansion, but were able to calculate the single-particle Green's function of a pure, $l = \infty$, type-II superconductor to a high degree of accuracy by using the periodicity of the known solution of the order parameter in fields near H_{c2}. In their original paper BPT calculated the angular-dependent density of states $N(\omega, \theta)$ (here θ is the angle between the direction of propagation of a quasiparticle $\vec{p}$ and the applied magnetic field), which was found to be BCS-like only if $\theta = 0$ and is gapless in all other directions. The BPT calculation was later generalized to the case of finite mean free path by Brandt[10] who used the Green's function to calculate the field dependence of the magnetization[11] and order parameter[12] near H_{c2}, obtaining good agreement with experiment.

It is desirable to have a first-principles theory of transport phenomena in clean type-II superconductors. Encouraged by the success of the calculation of equilibrium properties by BPT we have generalized the theory in such a way that it is now possible within this framework to determine the transport properties near H_{c2}. In order to facilitate comparison of theory with experiment, we have chosen, as an application of the theory, to calculate the change in attenuation of high-frequency longitudinal sound near H_{c2},[13] since it is well known[14] that, in this case, the effect of dynamical fluctuations of the order parameter is negligible. It is a straightforward matter to determine the effect of fluctuations, if necessary, using the same method. There is, in fact, an example of such a calculation already in the literature, namely, the work of Hibler and Cyrot[15] on the electrical conductivity near H_{c2}. Disagreement between experiment and theory in this case is probably due, as the authors point out, to the fact that they have implicitly assumed that $\omega\tau \gg 1$ (ω is the frequency of the wave) whereas in current experiments, in the megahertz region, $\omega\tau \ll 1$. It will be clear from the work of this paper that, under these conditions, it is essential to retain a finite mean free path even if $l \gg \xi_0$.

In Sec. II we develop the theory of longitudinal sound in clean type-II superconductors using methods similar to those used by BPT. In Sec. III the attenuation coefficient is calculated explicitly, and simple analytic expressions are derived in several limiting cases.

II. GENERAL THEORY

In this section we outline the theory of attenuation of a longitudinal sound wave in clean type-II superconductors near the upper critical field H_{c2}. It will be assumed throughout that the vortex lattice produced by the nearly uniform magnetic field near H_{c2} is static. The effect of fluctuations of the vortex lattice on the longitudinal attenuation has already been calculated by Caroli and Maki[14] and shown to be negligible.

It is well known[16] that if $\omega_s < \pi T_c$ and $ql \gg 1$, then the attenuation coefficient is simply related to the density-density correlation function; ω_s and $\vec{q}$ are the frequency and wave vector of the sound wave, respectively. As the sound wavelength is much larger than the intervortex spacing, it is sufficient to determine the spatial average of the correlation function. The attenuation rate is then determined by

$$\alpha_L = \mathrm{Re}\left(\frac{q^2}{i\omega\rho_{\mathrm{ion}} v_s}\right)\left(\frac{p_F^2}{3m}\right)^2 \langle [n, n]\rangle (\vec{q}, \omega_s) . \quad (1)$$

In Eq. (1) p_F is the Fermi momentum and v_s the velocity of sound. The function $\langle [n, n]\rangle, (\vec{q}, \omega_s)$, which is the Fourier transform of the volume average of the density-density correlation function, will be obtained by analytic continuation of the thermal product $\langle [n, n]\rangle (\vec{q}, \omega_0)$ from the set of discrete points $i\omega_0 = i2n\pi T$ to $z = \omega_s - i\delta$.

The thermal product must be evaluated for the mixed state of an impure type-II superconductor near the upper critical field. However, for simplicity, we limit the initial discussion to the case of a pure type-II superconductor. Since near H_{c2} the magnetic field inside the superconductor is nearly constant, it can be approximated by its spatial average $\vec{B}$. We choose $\vec{B}$ in the z direction and represent it by the gauge $\vec{A} \doteq (0, Bx, 0)$. The thermal product is decomposed by means of the Gorkov[17] factorization; to this end, it is convenient to introduce Green's functions defined by the equations

$$G^B(\vec{r}_2, \vec{r}_1, \omega) = \exp\left(ie\int_{r_2}^{r_1} \vec{A}\cdot d\vec{s}\right) G(\vec{r}_2, \vec{r}_1, \omega) , \quad (2)$$

$$G^0(\vec{r}_2 - \vec{r}_1, \omega) = \exp\left(ie\int_{r_2}^{r_1} \vec{A}\cdot d\vec{s}\right) \tilde{G}^0(\vec{r}_2, \vec{r}_1, \omega) . \quad (3)$$

Here $G(\vec{r}_2, \vec{r}_1, \omega) = -\langle T[\psi(\vec{r}_2, t)\psi^\dagger(\vec{r}_1, 0)]\rangle_\omega$ is the Green's function for a superconductor in the presence of a constant magnetic field, and $\tilde{G}^0$ is the normal-state Green's function in the presence of a constant magnetic field: The line integrals in Eqs. (2) and (3) are taken along a straight line connecting $\vec{r}_1$ and $\vec{r}_2$. $G^0(\vec{r}_2 - \vec{r}_1, \omega)$ is well approximated by the normal-state Green's function in zero field.[18]

Recently BPT[9] have shown that $G^B(\vec{r}_2, \vec{r}_1, \omega)$ has the important property of being periodic in the sum of its spatial variables, with the periodicity of the flux line lattice. This is easily shown if we note that the Gorkov[18] integral equation for G can be written

$$G^B(\vec{r}_1, \vec{r}_2, \omega) = G^0(\vec{r}_1 - \vec{r}_2, \omega) - \int d^3l\, d^3m\, G^0(\vec{r}_1 - \vec{l}, \omega) \times V(\vec{l}, \vec{m}) G^0(\vec{m} - \vec{l}, -\omega) G^B(\vec{m}, \vec{r}_2, \omega) , \quad (4)$$

where

$$V(\vec{l}, \vec{m}) = \exp\left(2ie\int_l^m \vec{A}\cdot d\vec{s}\right)\Delta(\vec{l})\Delta^*(\vec{m}) . \quad (5)$$

It was assumed by BPT that near H_{c2} the order parameter is given by the Abrikosov solution of the Ginzburg-Landau equations,[19] that is,

$$\Delta(\vec{r}) = \sum_k C_k e^{iky} e^{-eB(x-k/2eB)^2} ; \quad (6)$$

here $k = (4\pi eB)^{1/2} n$ and n is an integer and $\hbar = c = 1$ throughout. For simplicity we have used the form of $\Delta(\vec{r})$ corresponding to a square lattice; as we shall see, the results of this paper are independent of the symmetry of the flux line lattice. Using Eq. (6) it is easy to see that $V(\vec{l}, \vec{m})$ has the periodicity of the vortex lattice with respect to the sum coordinate. Therefore, from Eq. (4), G^B has the same periodicity.

It is instructive at this point to give a brief outline of the approximate calculation of G^B. Using the periodicity of G^B, Eq. (4) is Fourier transformed to give

$$G^B_\omega(\vec{p} - \tfrac{1}{2}\vec{k}, -\vec{k}) = \delta_{k,0} G^0_\omega(\vec{p}) - G^0_\omega(\vec{p} - \vec{k}) \times \sum_{k'} G^B_\omega(\vec{p} - \tfrac{1}{2}[\vec{k} + \vec{k}'], -\vec{k} - \vec{k}') \times \int \frac{d^3p'}{(2\pi)^3} V(\vec{p}', \vec{k}') G^0_{-\omega}(\vec{p} - \vec{p}' - \vec{k} - \tfrac{1}{2}\vec{k}') , \quad (7)$$

where $V(\vec{p}', \vec{k}')$, the Fourier transform of $V(\vec{l}, \vec{m})$, is given by

$$V(\vec{p}', \vec{k}') = \delta(p_z')\Delta^2\Lambda^2 2(2\pi)^2 \exp[-\Lambda^2(p_x'^2 + p_y'^2) + i\Lambda^2(-p_x'k_y' + p_y'k_x' + \tfrac{1}{2}k_x'k_y')] . \quad (8)$$

In Eq. (8), Δ^2 is the spatial average of the square of the absolute value of the order parameter and the length $\Lambda = (2eB)^{-1/2}$; $k = n(2\pi)^{1/2}\Lambda^{-1}$. We note that when $H \cong H_{c2}$, $\Lambda = \xi(T)$, whereas when $H = H_{c1}$, $\Lambda = \infty$. As the dominant contribution to most linear response functions comes from the Green's function $G^B(\vec{p}, 0)$, we will discuss the calculation of this function in detail. From Eq. (7) we have

$$G_\omega^B(\vec{p},\,0)=G_\omega^0(\vec{p})-G_\omega^0(\vec{p})\sum_{k'}G_\omega^B(\vec{p}-\tfrac{1}{2}\vec{k}',\,-\vec{k}')\times\int\frac{d^3p'}{(2\pi)^3}\,V(\vec{p}',\,\vec{k}')G_{-\omega}^0(\vec{p}-\vec{p}'-\tfrac{1}{2}\vec{k}')\;. \quad (9)$$

The magnitude of the coupling between $G_\omega^B(\vec{p},\,0)$ and $G_\omega^B(p,\,k'=0)$ is determined by the integral

$$\int\frac{d^3p'}{(2\pi)^3}\,V(p',\,k'\neq 0)=\Delta^2\exp[-\Lambda^2(\tfrac{1}{4}k'^2-\tfrac{1}{2}ik_x'k_y')]\ll\Delta^2\;; \quad (10)$$

therefore, to a high degree of accuracy, all terms other than $k'=0$ may be neglected with the result

$$G_\omega^B(\vec{p},\,0)=\left[i\omega_n-\xi_p+\frac{i\pi^{1/2}\Delta^2}{k_cv_F\sin\theta}\,W\!\left(\frac{i\omega_n+\xi_p}{k_cv_F\sin\theta}\right)\right]^{-1}. \quad (11)$$

Here $\xi_p=p^2/2m-\mu$, θ is the angle between the direction of propagation of the quasiparticle and the static magnetic field, $k_c=1/\Lambda$, and the function W is given by

$$W(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z-t}\,dt\;. \quad (12)$$

When the quasiparticle momentum is parallel to $\vec{B}$, it sees a constant order parameter, and $G_\omega^B(|p|,\theta=0)$ reduces to the BCS Green's function with a gap $\Delta^2=\langle|\Delta|^2\rangle$, as we would expect; however, in all other directions the quasiparticles have a finite lifetime due to particle-hole scattering induced by the spatial variation of the order parameter.

Using these results, and

$$|G_\omega^B(\vec{p},\,0)|\gg|G_\omega^B(\vec{p},\,\vec{k}\neq 0)|\;, \quad (13)$$

which can be proved by similar arguments, the attenuation coefficient can be written

$$\alpha_L=\frac{q}{v_s^2\rho_{\rm ion}}\,{\rm Im}\left(\frac{p_F^2}{3m}\right)^2\int\frac{d\omega}{2\pi i}\,[f(\omega+\omega_s)-f(\omega)]\times\{\langle[n,n]\rangle(\vec{q},\,\omega+z,\,\omega+i\delta)-\langle[n,n]\rangle(\vec{q},\,\omega+z,\,\omega-i\delta)\}, \quad (14)$$

where

$$\langle[n,n]\rangle(\vec{q},\,\omega+z,\,\omega+i\delta)=2\int\frac{d^3p}{(2\pi)^3}\,G_{\omega+z}^B(\vec{p}+\vec{q},\,0)G_{\omega+i\delta}^B(\vec{p},\,0)\times\left(1-\int\frac{d^3p'}{(2\pi)^3}\,G_{\omega+z}^0(\vec{p}+\vec{q}-\vec{p}')V(p',\,0)G_{-(\omega+i\delta)}^0(\vec{p}-\vec{p}')\right) \quad (15)$$

and $z=\omega_s-i\delta$.

It is now straightforward to generalize the theory to describe an impure type-II superconductor; in Eq. (14), we simply replace $\omega\pm i\delta$ by $\omega\pm i/2\tau$ and Δ by $\tilde{\Delta}=J\Delta$, where J is the Helfand-Werthamer[20] renormalization factor given by

$$J=\left[1-\frac{2}{k_cl}\int_0^\infty dt\,e^{-t^2}\tan^{-1}\!\left(\frac{it}{\gamma}\right)\right]^{-1}, \quad (16)$$

where $\gamma=(2i\omega_n/k_cv_F+i/k_cl)$. However, as we only consider the clean limit, where $J=1+O(\xi_0/l)$, this effect may be neglected.[21]

III. EVALUATION OF ATTENUATION COEFFICIENT

In this paper we limit our calculations of the attenuation, given, in general, by Eqs. (14) and (15), to the case of high-frequency sound; in particular we assume qv_F greater than the width of ${\rm Im}\,G^B \cong 1/\tau+\pi^{1/2}(\Delta^2/k_cv_F)$. In a pure type-II superconductor near H_{c2} this condition can be met if $\omega_s\gtrsim 10^9$ cps. This assumption enables us to make use of the transformation

$$\int_0^\infty p^2dp\int_0^\pi\sin\theta'd\theta'=\frac{m^2}{q}\int_{-\mu}^{\infty}d\xi_p\int_{\xi_p-v_Fq}^{\xi_p+v_Fq}d\xi_{p'}\;, \quad (17)$$

where $q^2=|p-p'|^2=|p^2+p'^2-2pp'\cos\theta'|$. As the dominant contribution to the integral comes from $p\cong p_F$, we may extend the lower limit of integration in the first integral to $-\infty$. Then under the conditions given above, the limits on the second integral may also be extended to ∞. The energy integrals are now easily performed by contour integration if we recall that

$$G_{\omega\pm i/2\tau}^B(\vec{p},\,0)=\left[\omega\pm\frac{i}{2\tau}-\xi_p+\frac{i\pi^{1/2}\Delta^2}{k_cv_F\sin\theta}\,W\!\left(\frac{\omega\pm i/2\tau+\xi_p}{k_cv_F\sin\theta}\right)\right]^{-1} \quad (18)$$

has only a simple pole in the (upper, lower) half-plane, respectively, at

$$\xi_0 = \omega \pm \frac{i}{2\tau} + \frac{i\pi^{1/2}\Delta^2}{k_c v_F \sin\theta} W\left(\frac{\omega \pm i/2\tau + \xi_0}{k_c v_F \sin\theta}\right), \quad (19)$$

and we obtain

$$\alpha_L = \frac{m^2}{v_s^2 \rho_{\text{ion}}} \left(\frac{p_F^2}{3m}\right)^2 \int \frac{d\phi}{2\pi} \int \frac{d\omega}{2\pi} [f(\omega) - f(\omega+\omega_s)] I(\omega, \theta), \quad (20)$$

where

$$I(\omega, \theta) = [2\text{Re}K(z_0) - 1] \times [2\text{Re}\,K(z_0^+) - 1] - 2\left(\frac{\Delta}{k_c v_F \sin\theta}\right)^2 \times \text{Re}\left(K(z_0)K(z_0^+)\frac{i\pi^{1/2}W(z_0) - i\pi^{1/2}W(z_0^+)}{z_0 - z_0^+} + K^*(z_0)K(z_0^+)\frac{i\pi^{1/2}W(z_0^*) - i\pi^{1/2}W(z_0^+)}{z_0^* - z_0^+}\right), \quad (21)$$

$$K(z_0) = \left[1 - i\pi^{1/2}\left(\frac{\Delta}{k_c v_F \sin\theta}\right)^2 W'(z_0)\right]^{-1}, \quad (22)$$

$$z_0 = (\xi_0 + \omega + i/2\tau)/k_c v_F \sin\theta, \quad (23)$$

$$z_0^+ = z_0(\omega + \omega_s), \quad (24)$$

$$\sin\theta = (1 - \sin^2\alpha \cos^2\phi)^{1/2}, \quad (25)$$

and α is the angle between the direction of propagation of the sound wave and the magnetic field. In deriving Eqs. (17)–(25) we have also made use of the fact that if $qv_F \gg 1/\tau + \pi^{1/2}(\Delta^2/k_c v_F)$ only those electrons moving perpendicular to the sound wave contribute to the attenuation. At arbitrary temperature and frequency the attenuation can only be evaluated numerically. Further analytic progress is only possible if we make the additional assumption that $\omega_s \ll$ the width of $\text{Im}\,G^B(\vec{p}, 0)$, that is, $\omega_s \lesssim 1/\tau + \pi^{1/2}(\Delta^2/k_c v_F)$. As we have already assumed $\omega_s > (v_s/v_F)[1/\tau + \pi^{1/2}(\Delta^2/k_c v_F)]$ and as in niobium $v_F/v_s \cong 10^2$, we should not expect to obtain better than qualitative agreement with experiment. We feel, however, that the simple analytic results derived in this manner describe the main features of the change in attenuation near H_{c2}.

Keeping only lowest-order terms in ω_s, we find

$$I(\omega, \theta) = [2\text{Re}K(z_0) - 1]^2 - 2\left(\frac{\Delta}{k_c v_F \sin\theta}\right)^2 \times \text{Re}\left([K(z_0)]^2 i\pi^{1/2} W'(z_0) + \frac{|K(z_0)|^2 \text{Im}\, i\pi^{1/2}W(z_0)}{1/k_c l \sin\theta + (\Delta/k_c v_F \sin\theta)^2 \text{Im}\, i\pi^{1/2}W(z_0)}\right). \quad (26)$$

In the low-temperature limit, it is sufficient to replace $f(\omega) - f(\omega + \omega_s)$ by $\omega_s \delta(\omega)$. The integration over ω is then trivial to carry out with the result that z_0 of Eq. (23) is given by

$$z_0 = i/k_c l \sin\theta + i\pi^{1/2}(\Delta/k_c v_F \sin\theta)^2 W(z_0). \quad (27)$$

If we now use the relationship[22]

$$W(z) = e^{-z^2}\,\text{erfc}(-iz) \quad (28)$$

for z in the upper half-plane, where

$$\text{erfc}(-iz) = (2/\pi^{1/2}) \int_{-iz}^{\infty} e^{-t^2} dt, \quad (29)$$

and notice that if z is pure imaginary ($z = iy$) then

$$W(iy) = (2/\pi^{1/2}) e^{y^2} \int_y^{\infty} e^{-t^2} dt, \quad (30)$$

it immediately follows that the solution of Eq. (27), $z_0 = ix_0/\sin\theta$, is pure imaginary. Further, if we make use of the inequality

$$2[y + (y^2+2)^{1/2}]^{-1} \leq \pi^{1/2} W(iy) \leq 2[y + (y^2+4)^{1/2}]^{-1}, \quad (31)$$

which may be written

$$\sqrt{\pi}\, W(iy) = 2\{y + [y^2 + \alpha(y)]^{1/2}\}^{-1}, \quad (32)$$

where[23]

$$2 \geq \alpha(y) \geq 4/\pi$$

Eq. (27) can be solved with the result that

$$x_0 = 1/k_c l + r, \quad (33)$$

where

$$r = \frac{2(\Delta/k_c v_F)^2}{\alpha \sin^2\theta + 4(\Delta/k_c v_F)^2} \times \left\{\left[\frac{1}{(k_c l)^2} + \alpha \sin^2\theta + 4\left(\frac{\Delta}{k_c v_F}\right)^2\right]^{1/2} - \frac{1}{k_c l}\right\}. \quad (34)$$

In this limit it is also possible to show that $K(z_0)$ is real:

$$K\left(\frac{ix_0}{\sin\theta}\right) = \left[1 + 2\left(\frac{\Delta}{k_c v_F}\right)^2 \frac{\alpha}{4(\Delta/k_c v_F)^2 + \alpha \sin^2\theta}\right.$$

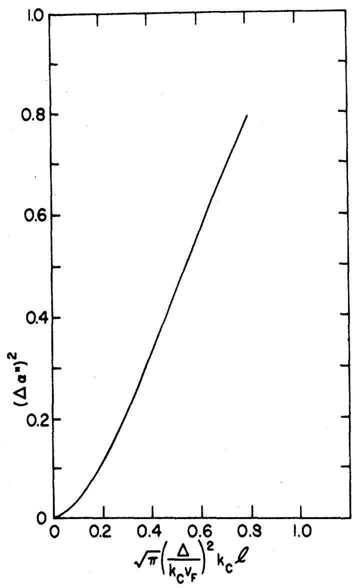

FIG. 1. Low-frequency, $\omega_s \ll (1/\tau + \Delta^2/k_c v_F)$, ultrasonic attenuation in clean type-II superconductors near H_{c2}. $(\Delta\alpha'')^2$ as a function of $\Gamma = \pi^{1/2}(\Delta/k_c v_F)^2 k_c l$.

$$\times \left[\frac{[1/(k_c l)^2 + \alpha \sin^2\theta + 4(\Delta/k_c v_F)^2]^{1/2} - 1/k_c l}{[1/(k_c l)^2 + \alpha \sin^2\theta + 4(\Delta/k_c v_F)^2]^{1/2} + 1/k_c l}\right]^{-1}, \tag{35}$$

and therefore

$$I(0,\theta) - 1 = 2K\left(\frac{ix_0}{\sin\theta}\right)\left[K\left(\frac{ix_0}{\sin\theta}\right) - 1\right]$$

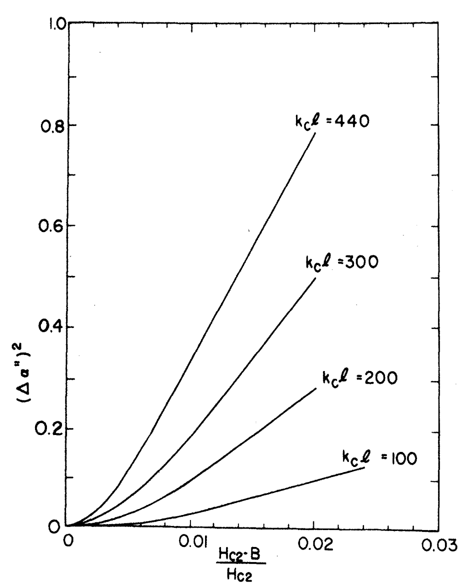

FIG. 2. Low-frequency ultrasonic attenuation in clean type-II superconductors near H_{c2} at $T = 1.51\,°K$. $(\Delta\alpha'')^2$ as a function of $(H_{c2} - B)/H_{c2}$ for different values of the mean free path. The values of $k_c l$ quoted in the figure are taken at $H = H_{c2}$.

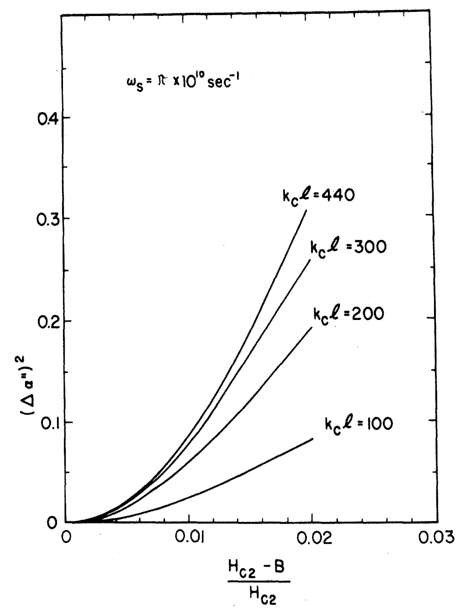

FIG. 3. Ultrasonic attenuation in clean type-II superconductors near H_{c2} at $\omega_s = \pi \times 10^{10}\ \mathrm{sec}^{-1}$ and $T = 1.51\,°K$, $(\Delta\alpha'')^2$ as a function of $(H_{c2} - B)/H_{c2}$ for different values of the mean free path.

$$-2K^2\left(\frac{ix_0}{\sin\theta}\right)\frac{k_c l r}{1 + k_c l r}. \tag{36}$$

This expression can be used to compute the attenuation at any angle of propagation. We will only

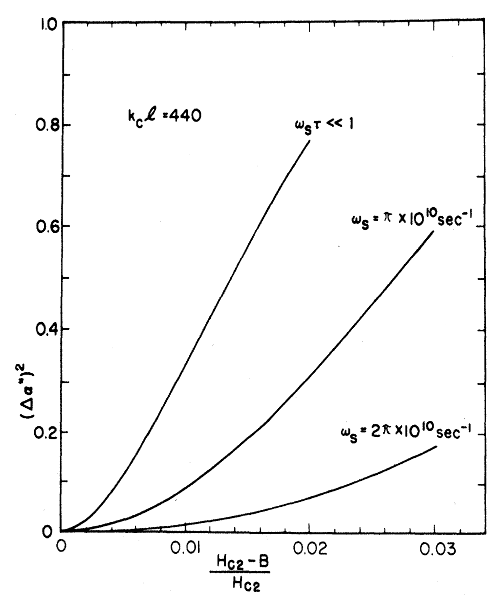

FIG. 4. Ultrasonic attenuation in clean type-II superconductors near H_{c2} at $k_c(H_{c2})\,l = 440$ and $T = 1.51\,°K$. $(\Delta\alpha'')^2$ as a function of $(H_{c2} - B)/H_{c2}$ for different values of the frequency.

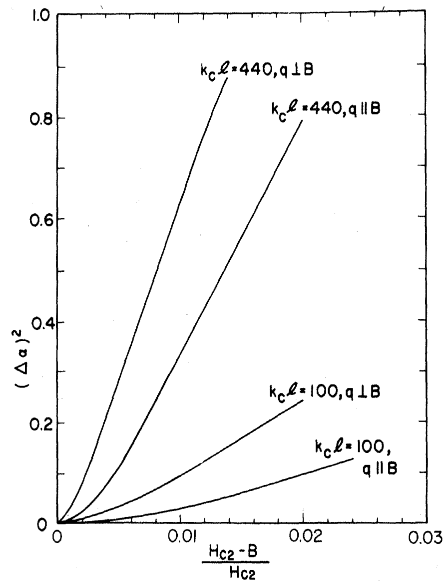

FIG. 5. Low-frequency ultrasonic attenuation in clean type-II superconductors near H_{c2} at $T=1.51\,°\mathrm{K}$, $(\Delta\alpha^{\parallel})^2$ and $(\Delta\alpha^{\perp})^2$ as function of $(H_{c2}-B)/H_{c2}$ at $k_c(H_{c2})\,l=100$ and $k_c(H_{c2})\,l=440$.

discuss in detail the cases of propagation parallel and perpendicular to $\vec{B}$.

When the sound wave propagates parallel to the magnetic field, $\sin\theta=1$, and it is easy to see that as $k_c l \gg 1$,

$$r \cong 2\left(\frac{\Delta}{k_c v_F}\right)^2\left[\alpha + 4\left(\frac{\Delta}{k_c v_F}\right)^2\right]^{-1/2}, \tag{37}$$

$$K(ix_0) = 1 + 2(\Delta/k_c v_F)^2 . \tag{38}$$

Using the fact that in this case $|x_0| \ll 1$ and $\alpha(x_0) = 4/\pi$, we find

$$\Delta\alpha^{\parallel} = \frac{\alpha_s^{\parallel}}{\alpha_n} - 1 = -2\frac{\pi^{1/2}(\Delta/k_c v_F)^2 k_c l}{1+\pi^{1/2}(\Delta/k_c v_F)^2 k_c l} . \tag{39}$$

Thus we have obtained the result that, in pure type-II superconductors near H_{c2}, $\Delta\alpha^{\parallel}$ is a universal function of the parameter $\Gamma = \pi^{1/2}(\Delta/k_c v_F)^2 k_c l$. A graph of this function is shown in Fig. 1. In Fig. 2 we plot $(\Delta\alpha^{\parallel})^2$ as a function $H_{c2}-B$ for field values close to H_{c2}, and different values of the mean free path. As can be seen from the figure $(\Delta\alpha^{\parallel})^2$ could be considered to be linear in $H_{c2}-B$ over a narrow field range; and is also strongly mean free path dependent. Both these effects are consistent with experiment. Care, however, should be used in comparing the predictions of Eq. (39) with current experiments. First, at present, experiments have only been carried out under the condition $ql<1$; we expect, however,[24] that the main features of the attenuation should be the same in this case. Second, as we have already pointed out, the magnitude of the attenuation predicted by Eq. (39) should not be interpreted literally. For example, as can be seen by comparison with an exact numerical calculation, the magnitude of $\Delta\alpha^{\parallel}$ predicted by Eq. (39) is too large. In Fig. 3 we present the results of numerical calculations of $(\Delta\alpha^{\parallel})^2$ for a frequency of $\pi\times 10^{10}\,\mathrm{sec}^{-1}$ at a temperature of $T=1.51\,°\mathrm{K}$. It can be seen that the trend, as a function of mean free path, and shape of the curves, as a function of $H_{c2}-B$, given in Fig. 2 are retained in the exact calculation; however, the magnitude of the attenuation is much smaller. It can also be seen from Fig. 3 that the attenuation at a given frequency becomes less purity dependent as the mean free path increases, which has also been observed experimentally.[7] Finally, in Fig. 4 we see that in the high-frequency limit the attenuation is strongly frequency dependent. In all of the numerical calculations the physical parameters used were Δ^2, which was taken from Eilenberger's paper,[25] $H_{c2}(1.51\,°\mathrm{K}) = 3625$ G,[26] $v_F = 3.0\times10^7$ cm/sec, and the density of states[27] $N(0) = 6.0\times10^{34}$ erg/cm^3.

The final example we consider is when the sound wave propagates perpendicular to $\vec{B}$. In this case, provided $\Delta\tau \gg 1$, which at $k_c l = 100$, for example, is valid if $(H_{c2}-B)/H_{c2} \gtrsim 10^{-3}$, we have

$$r \cong 2\left(\frac{\Delta}{k_c v_F}\right)^2\left[\alpha\sin^2\theta + 4\left(\frac{\Delta}{k_c v_F}\right)^2\right]^{-1/2}, \tag{40}$$

$$K\left(\frac{ix_0}{\sin\theta}\right) = 1 - 2\left(\frac{\Delta}{k_c v_F}\right)^2 \alpha \Bigg/ \left[\alpha\sin^2\theta + 2\left(\frac{\Delta}{k_c v_F}\right)^2(2+\alpha)\right], \tag{41}$$

and therefore

$$\Delta\alpha^{\perp} = -\frac{1}{2\pi}\int_0^{2\pi} d\phi\left(\frac{4(\Delta/k_c v_F)^2\alpha}{\alpha\sin^2\phi + 2(\Delta/k_c v_F)^2(2+\alpha)} + \frac{4(\Delta/k_c v_F)^2 k_c l}{\alpha^{1/2}|\sin\phi| + 2(\Delta/k_c v_F)^2 k_c l}\right), \tag{42}$$

that is,

$$\Delta\alpha^{\perp} = -2\frac{\Delta}{k_c v_F} - \frac{4}{\pi}\frac{2\Gamma}{\Gamma^2-1}\tan^{-1}\left(\frac{\Gamma-1}{\Gamma+1}\right)^{1/2} . \tag{43}$$

It is interesting to note that in this geometry the expression for the attenuation contains terms proportional to Δ and therefore $(H_{c2}-B)^{1/2}$; however, as the second term in the expression dominates except at fields arbitrarily close to H_{c2} it is unlikely that this behavior can be detected experimentally. We might also point out that if $\Gamma \cong 1$, then Eq. (43) becomes

$$\Delta\alpha^{\perp} = -(4/\pi)[2\Gamma/(1+\Gamma)] = (4/\pi)\,\Delta\alpha^{\parallel} . \quad (44)$$

Equation (44) exhibits explicitly the anisotropy in the attenuation. This is easily understood if we recall that the quasiparticle propagator depends strongly on the direction of propagation of the quasiparticle. In Fig. 5 the values of $(\Delta\alpha^{\parallel})^2$ and $(\Delta\alpha^{\perp})^2$ are compared for two different mean free paths.

ACKNOWLEDGMENTS

The authors would like to thank Dr. M. Cyrot, Dr. E. Forgan, and Dr. K. Maki for several helpful discussions of points contained in this paper.

*Work supported in part by the Advanced Projects Agency and the National Science Foundation.

[1]K. Maki, in *Superconductivity*, edited by R. Parks (Marcel Dekker, New York, 1969), Chap. 18.

[2]M. Cyrot and K. Maki, Phys. Rev. 156, 433 (1967).

[3]K. Maki, Phys. Rev. 156, 437 (1967).

[4]K. Maki, Phys. Rev. 158, 397 (1967).

[5]F. B. McLean and A. Houghton, Phys. Rev. 157, 350 (1967); K. Maki and P. Fulde, Solid State Commun. 5, 21 (1967); C. Caroli and M. Cyrot, Physik Kondensierten Materie 4, 285 (1965).

[6]R. Kagiwada, M. Levy, I. Rudnick, H. Kagiwada, and K. Maki, Phys. Rev. Letters 18, 835 (1967).

[7]E. M. Forgan and C. E. Gough, Phys. Letters 26A, 602 (1968).

[8]W. F. Vinen, E. M. Forgan, C. E. Gough, and M. J. Hood, Physica (to be published).

[9]U. Brandt, W. Pesch, and L. Tewordt, Z. Physik 201, 209 (1967).

[10]U. Brandt, Phys. Letters 27A, 645 (1968).

[11]U. Brandt, Phys. Letters 29A, 568 (1969).

[12]U. Brandt, thesis (unpublished).

[13]A preliminary report on this theory was given by H. A. Cerdeira, W. N. Cottingham, and A. Houghton, Solid State Commun. 7, 1409 (1969); J. Phys. Chem. Solids 31, 1681 (1970).

[14]C. Caroli and K. Maki, Phys. Rev. 159, 316 (1967).

[15]W. D. Hibler, III and M. Cyrot, Phys. Rev. B 1, 3007 (1970).

[16]L. P. Kadanoff and I. I. Falko, Phys. Rev. 136, A1170 (1964).

[17]L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. 34, 735 (1958) [Sov. Phys. JETP 7, 505 (1958)].

[18]L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. 36, 1918 (1959) [Sov. Phys. JETP 9, 1364 (1959)].

[19]A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. 32, 1442 (1957) [Sov. Phys. JETP 5, 1174 (1957)].

[20]E. Helfand and N. R. Werthamer, Phys. Rev. 147, 288 (1966).

[21]It should be pointed out that at $T \simeq T_c$, $k_c \to 0$, and the vertex corrections can no longer be neglected.

[22]W. Gautschi, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, Natl. Bur. Stds. Applied Mathematics Series No. 55 (U. S. GPO, Washington, D. C., 1964).

[23]We note that by using the asymptomatic expansion of the function $W(z)$ of imaginary argument it can be verified that $\alpha(0) = 4/\pi$ and $\alpha(\infty) = 2$.

[24]A. Houghton and K. Maki Phys. Rev. B (to be published).

[25]G. Eilenberger, Phys. Rev. 153, 1584 (1967).

[26]T. McConville and B. Serin, Phys. Rev. 140, A1169 (1965).

[27]H. A. Leupold and H. A. Boorse, Phys. Rev. 134, A1322 (1964).