Optics Communications 88 (1992) 258–272
North-Holland

OPTICS
COMMUNICATIONS

Full length article

Collective effects in an atomic layer near a phase conjugated mirror

C.L. Pando L. and Hilda Cerdeira
International Center for Theoretical Physics (ICTP), P.O. Box 586, I-34100 Trieste, Italy

Received 15 July 1991

In the present paper we study using the Maxwell–Bloch equations the interaction between a phase conjugated mirror with a thin film of two-level atoms, which in turn are driven by an external coherent field. We find several bifurcation phenomena in the coherent dynamics as well as in the steady-state regime of this system. In particular we find that when the system is in its steady-state regime due to incoherent relaxation, the population inversion of the thin-film atoms may remain constant or even decrease as the external field intensity increases.

1. Introduction

There is a growing interest in the study of thin films of two-level atoms since it is a model that may reproduce many collective phenomena in quantum optics such as superradiance [1,2] or optical bistability [3] without invoking the spatial mean-field approximation. For this model several papers were published in the context of coherent phenomena and open dissipative systems. Nonlinear surface waves [4], nonlinear reflection of ultrashort light pulses [5] among other effects belong to those coherent phenomena described within the framework of the thin-film model. It is worth to point out that the thin-film model has a direct analogy with the semi-classical Dicke model. In the former model the electromagnetic field acting as a feedback arises as a result of the surface current in the boundary conditions, while in the latter model the feedback is caused by the reaction field [6]. In its simplest versions dipole–dipole interactions are not considered in both models. On the other hand, effects such as optical bistability and self-pulsations may be described within the thin-film model [7–10]. In this paper a basic new ingredient we add in the context of the thin-layer models is that we are assuming that a phase conjugated mirror (PCM) is placed in front of a thin layer of two-level atoms, which in turn is driven by an external coherent field. We assume that the transition frequency of the atoms of the layer is at exact resonance with the frequencies of the external field and the PCM pump fields. Our model has an exact solution and in that sense it is useful to obtain insight into the physics of related problems.

Experimental and theoretical research in phase conjugation is a rich field in quantum optics [11]. Nevertheless the study of decay processes in quantum mechanical systems such as two-level atoms interacting with a PCM is a matter of recent interest [12–18]. For our work, refs. [12–18] are relevant in two aspects. Firstly, we assume that our system of two-level atoms is such that the PCM induced decay may be neglected (see appendix). Secondly, we further study the coherent dynamics of our system on the basis of a method devised by Pando [17] to analyze the coherent interaction between a PCM and a point sample of two-level atoms.

Another way to visualize our system is the following. We are dealing with a "Fabry-Perot resonator" where one of the mirrors is the PCM and the other one is the thin film of two-level atoms. The spacer filling this resonator has negligible loses for the intracavity plane waves. The properties of this resonator may be changed not only through the PCM parameters but also by varying the external coherent field. From this perspective, to our knowledge, this is the first time that this system is studied in the literature [11].

Related systems to ours are resonators consisting of one or two PCM, filled with or without absorbing or dispersive media which may be pumped by an external field [11]. The model equations describing these systems differ from ours, since we are dealing with the bad cavity limit (see eq. (10)).

In our system we study coherent phenomena as well as the stationary regime due to incoherent damping processes. The coherent phenomena are described using an analogy with the motion of an imaginary particle in a potential well [19]. On the other hand with regard to the stationary regime we analyze the case when the external coherent field is absent or present for two values of the relative phase between the external field and the PCM intrinsic phase. To our knowledge most of our results are new and they are discussed briefly in section 5.

2. General theory

Let us consider a system consisting of two flat interfaces which are the boundaries of a semi-infinite phase conjugated mirror, a dielectric spacer with dielectric constant ϵ_2 having thickness l and a semi-infinite dielectric substrate with dielectric constant ϵ_1. A thin layer of two-level atoms with thickness $\delta \ll \lambda$, where λ is the wavelength of the emitted light by the atoms in vacuum, is localized at the interface ($z=0$) between the linear media with dielectric constants ϵ_1 ($z<0$) and ϵ_2 ($0<z<l$), $l \gg \lambda$. l has the order of magnitude of the distance from the atomic dipole to its nearest radiation zone, so that retardation effects are negligible. A coherent applied field is incident perpendicularly upon the interfaces from medium 1 ($z<0$). The field is a plane wave polarized along the x-axis tuned at exact resonance with the atoms. Taking into account the presence of the thin polarizing layer of two-level atoms, the boundary conditions at $z=0$ are

$$E_x(0^+,t)-E_x(0^-,t)=0\,,\quad H_y(0^+,t)-H_y(0^-,t)=-\frac{4\pi}{c}\frac{\partial P_x}{\partial t}\,. \tag{1}$$

Here $E_x(z,t)$ and $H_y(z,t)$ are the components of the electromagnetic field [20]. P_x is the surface polarization of the atomic thin film. The boundary conditions at $z=l$ will be written below. The electric fields in both dielectric media and the polarization have the form:

for $z<0$

$$E_x^{(1)}(z,t)=(1/\sqrt{2})\{E_0(z,t)\exp[\mathrm{i}(k_1z-\omega t)]+E_r(z,t)\exp[-\mathrm{i}(k_1z+\omega t)]+\text{c.c.}\}\,,$$

for $0<z<l$

$$E_x^{(2)}(z,t)=(1/\sqrt{2})\{E_f(z,t)\exp[\mathrm{i}(k_2z-\omega t)]+E_b(z,t)\exp[-\mathrm{i}(k_2z+\omega t)]+\text{c.c.}\}$$

and

$$P_x=(1/\sqrt{2})[P\exp(-\mathrm{i}\omega t)+\text{c.c.}]\,, \tag{2}$$

where $E_0(z,t)$, $E_r(z,t)$, $E_f(z,t)$, $E_b(z,t)$ and P are the slowly varying envelopes of the incident, reflected, forward, backward and polarization fields respectively. $k_i=\omega\sqrt{\epsilon_i}$, $i=1,2$.

At the interface $z=l$, the phase conjugated mirror produces a reflected wave E_{PCM} whose amplitude is defined as follows [11]:

$$E_{\mathrm{PCM}}=E_b\exp(-\mathrm{i}k_2l)=\mu(E_f\exp(\mathrm{i}k_2l))^*\,, \tag{3}$$

where $\mu=|\mu|\exp(\mathrm{i}\psi)$, $|\mu|$ stands for the PCM gain coefficient. ψ is the PCM intrinsic phase [11]. Here we consider that the PCM is produced by a four-wave mixing process. Throughout this paper we will consider that the frequency of the incident and reflected photons from the PCM equals the frequency of the PCM pump photons. From now on we will not write the spatial and temporal dependence of the field envelopes.

As a result of eq. (3), we have now only two unknown fields, E_r and E_f. We replace eqs. (2) and (3) into the boundary conditions (1) using the first Maxwell equation:

$$\mathrm{rot}\,\boldsymbol{E} = -(1/c)\,\partial \boldsymbol{H}/\partial t\,,$$

where $\boldsymbol{E}=(E_x, 0, 0)$ and $\boldsymbol{H}=(0, H_y, 0)$. The first Maxwell equation enables us to express H_y in terms of E_x. Neglecting derivatives of the smooth envelopes, we find at $z=0$, the following equations:

$$E_r - E_f - \mu E_f^* = -E_0\,,$$

$$k_1 E_r + k_2(E_f - \mu E_f^*) = k_1 E_0 + 4\pi i(\omega/c)^2 P\,. \tag{4}$$

Notice that the fields E_r and E_f are complex quantities. Using the complex conjugate equations corresponding to eqs. (4) one finds the total field at $z=0$:

$$E_x^{(1)} = (1/\sqrt{2})(E^{(+)}+E^{(-)}) = (1/\sqrt{2})(E_0+E_r)\exp(-i\omega t)+\text{c.c.}$$
$$= (1/\sqrt{2}A)\{[k_1+k_2+|\mu|^2(k_2-k_1)][2k_1E_0+4\pi i(\omega/c)^2P]$$
$$+2k_2\mu[2k_1E_0^* - 4\pi i(\omega/c)^2P^*]\}\exp(-i\omega t)+\text{c.c.}\,, \tag{5}$$

where

$$A = (k_1+k_2)^2 - |\mu|^2(k_2-k_1)^2\,.$$

The case that will be studied in this paper corresponds to $k_1=k_2=k$. As a result the electric field at the atomic layer position reduces to

$$E_0+E_r = (2\pi i/k)(\omega/c)^2(P-\mu P^*)+E_0+\mu E_0^*\,. \tag{6}$$

Eq. (6) must be coupled with the Bloch equations. The thin atomic layer is modeled as an ensemble of two-level atoms where the local-field correction is not considered. We assume that our atoms are so strongly coupled to a nonradiative bath, that we may neglect incoherent radiative damping processes. The idea here is to find a system in which one can neglect the PCM induced decay. The latter may also be achieved by a sample of two-level atoms of finite dimensions where transverse and propagation effects may be neglected, and where atoms relax radiatively. In the appendix a discussion is given for our model that shows why the PCM induced decay may be neglected. In addition we consider dephasing mechanisms.

The self-consistent Bloch equations are given by

$$\mathrm{d}D/\mathrm{d}t = -\gamma_\parallel(D+1)+(i\chi E^{(+)}\alpha^*+\text{c.c.})\,,\quad \mathrm{d}\alpha/\mathrm{d}t = -(\gamma_\perp+i\omega)\alpha - i\chi E^{(+)}D\,. \tag{7}$$

Here D is the population inversion, α is the atomic polarization, $\gamma_\parallel$ and $\gamma_\perp$ are the longitudinal and the transverse relaxation rates respectively, which model the atomic relaxation. χ is the atom–field coupling constant. The atomic transition frequency is at exact resonance with that of the external field as well as with the frequency of the PCM pump fields. The slowly varying polarization P for homogeneously broadened atoms may be written as follows:

$$P = d_{12} n\alpha \exp(i\omega t)\,. \tag{8}$$

Here d_{12} is the dipole moment of the transition and n is the surface density of the two-level atoms.

We write the external coherent field at $z=0$ and the atomic polarization as follows:

$$E_0 = |E_0|\exp(i\varphi)\,,\quad \alpha = \tfrac{1}{2}(X+iY)\exp(i\phi)\exp(-i\omega t)\,. \tag{9}$$

After inserting eqs. (8) and (9) into eq. (7) and setting $\phi=\frac{1}{2}\psi$ we obtain

$$\mathrm{d}X/\mathrm{d}\tau = -\Gamma_\perp X + \Theta_x XD + \Omega_x D\,,\quad \mathrm{d}Y/\mathrm{d}\tau = -\Gamma_\perp Y + \Theta_y YD - \Omega_y D\,,$$

$$\mathrm{d}D/\mathrm{d}\tau=-\Gamma_{\parallel}(D+1)-\Theta_x X^2-\Theta_y Y^2-\Omega_x X+\Omega_y Y\,, \tag{10}$$

where

$$\tau=gt\,,\quad g=(2\pi\chi/k)(\omega/c)^2 d_{12} n\,,$$

$$\Theta_x=1-|\mu|\,,\quad \Theta_y=1+|\mu|\,,$$

$$\Omega_x=\Omega_0\Theta_x\sin(\varphi-\tfrac{1}{2}\psi)\,,\quad \Omega_y=\Omega_0\Theta_y\cos(\varphi-\tfrac{1}{2}\psi)\,,\quad \Omega_0=(2\chi/g)|E_0|\,.$$

$\Gamma_\perp$ and $\Gamma_\parallel$ stand for $\gamma_\perp/g$ and $\gamma_\parallel/g$ respectively, where $\gamma_\perp$ and $\gamma_\parallel$ are the transverse and longitudinal relaxation rates.

Let us point out that when $\mu=0$ eqs. (10) reduce to the equations describing optical bistability in Fabry-Perot resonators filled with two-level atoms in the bad cavity limit at exact resonance and in the mean-field approximation [21], ($\varphi=\psi=0$).

Eqs. (10), provided $\mu=0$, also describe the field–atom dynamics of thin layers of two-level atoms when the local-field correction (LFC) is not taken into account [22,23]. When the LFC is considered, a nonlinear term coupling both polarization components appears [5,7–10,24]. When $\Gamma_\perp=\Gamma_\parallel=0$ and $\Omega_x=\Omega_y=0$, similar equations were obtained for a point sample of two-level atoms in front of a PCM with finite dimensions [16].

Here we point out that the photons that arrive at the PCM front port are those that are coherently emitted by the atoms of the layer or the external field. In other words photons emitted by the atomic layer are produced by induced transitions due to the electric field at the atomic layer position. The emitted photons have frequency ω and therefore the PCM reflects photons at this frequency. Let us derive a general expression for the intensity of the reflected field. From eq. (6), it is easy to check using eqs. (8) and (9) that the reflected field E_r is given by

$$E_r=(\pi i d_{12} n/k)(\omega/c)^2\exp(\tfrac{1}{2}i\psi)\,(\Theta_x X+i\Theta_y Y)+|\mu|\,|E_0|\exp[i(-\varphi+\psi)]\,. \tag{11}$$

From eq. (11) we obtain the intensity of the reflected field I_r:

$$\begin{aligned}I_r&=(c/4\pi)E_r E_r^*\\&=(c/4\pi)[B\Theta_x X-|\mu|\,|E_0|\sin(\varphi-\tfrac{1}{2}\psi)]^2+(c/4\pi)[B\Theta_y Y-|\mu|\,|E_0|\cos(\varphi-\tfrac{1}{2}\psi)]^2\,,\end{aligned} \tag{12}$$

where

$$B=(\pi d_{12} n/k)(\omega/c)^2\,.$$

3. Coherent effects

Now we will describe phenomena that takes place when the interaction with the bath may be neglected, i.e. $\Gamma_\perp=\Gamma_\parallel=0$. As a result the Bloch vector conserves its magnitude, i.e. $X^2+Y^2+D^2=1$. The coherent interaction may be analyzed using an analogy with the motion of an imaginary particle in a potential well according to ref. [19]. In this limit the polarization components in eqs. (10) may be written as follows:

$$\mathrm{d}X/\mathrm{d}\xi=\Theta_x X+\Omega_x\,,\quad \mathrm{d}Y/\mathrm{d}\xi=\Theta_y Y-\Omega_y\,. \tag{13}$$

Here $\xi=\int_0^\tau D(q)\,\mathrm{d}q$. By solving eqs. (13) we find $X(\xi)$ and $Y(\xi)$ as functions of ξ. The whole dynamics of eqs. (10) in this limit is contained in the dynamics of the motion of the particle:

$$\tfrac{1}{2}(\mathrm{d}\xi/\mathrm{d}\tau)^2+[X^2(\xi)+Y^2(\xi)]=\tfrac{1}{2}\,. \tag{14}$$

Eq. (14) has been obtained from $X^2+Y^2+D^2=1$. $D=\mathrm{d}\xi/\mathrm{d}\tau$ is the velocity of the particle.

With these equations we may analyze and classify the motions that take place in the system. Properties of these equations are studied in ref. [19]. In eq. (14) when the imaginary particle moves to the right (left), the Bloch vector (X, Y, D) is in the upper (lower) hemisphere since $d\xi/d\tau=D$. If the imaginary particle reaches the turning point changing its direction of motion from right (left) to left (right), the Bloch vector reaches the equator $D=0$, passing from the upper (lower) to the lower (upper) hemisphere. In other words the system is being coherently deexcited (excited). Next we analyze two particular cases, where the external field is present or absent.

3.1. Zero external driving field case

Here $\Omega_x=\Omega_y=0$ in eqs. (13). We obtain

$$X=\rho\cos\nu\exp(\Theta_x\xi)\ ,\quad Y=\rho\sin\nu\exp(\Theta_y\xi)\ , \tag{15}$$

where

$$X(0)=\rho\cos\nu\ ,\quad Y(0)=\rho\sin\nu$$

and from eq. (14) we obtain

$$\tfrac{1}{2}(d\xi/d\tau)^2+\tfrac{1}{2}[\rho^2\cos^2\nu\exp(2\Theta_x\xi)+\rho^2\sin^2\nu\exp(2\Theta_y\xi)]=\tfrac{1}{2}\ . \tag{16}$$

This case has been studied in refs. [16,17]. In ref. [17] it was shown that a point sample of two-level atoms interacting with a PCM, a model that was originally proposed by Cook and Milonni [16], may be studied using an analogy with an imaginary particle. From ref. [17], we recall some results for completeness.

For $|\mu|<1$, the population inversion reaches the ground state as $\tau\to\infty$. The time when the atomic system decays faster occurs when the Bloch vector reaches the equator, $D=0$ [19]. Correspondingly the particle must reach the turning point ξ^* in eq. (16). Therefore for an initially excited system $D(0)>0$ nearly inverted, the delay time τ^* defined as the time when the atomic system decays faster, is given by

$$\tau^*=\int_0^{\xi^*} d\xi/\sqrt{1-V(\xi)}\ . \tag{17}$$

Here $V(\xi)$ is the expression in square brackets in eq. (16) (the potential well). Analogously for $|\mu|=1$, the atomic system approaches the asymptotic value $X=X(0)$, $Y=0$ and $D=-\sqrt{1-X^2(0)}$. For $|\mu|>1$, for any 0, $\pi/2$, π and $3\pi/2$, the motion is periodical ($\rho\neq 0$).

From eq. (12) we get the reflected intensity for this case:

$$I_r=(c/4\pi)B^2\rho^2[\Theta_x^2\cos^2\nu\exp(2\Theta_x\xi)+\Theta_y^2\sin^2\nu\exp(2\Theta_y\xi)]\ . \tag{18}$$

For $|\mu|<1$, the reflected light is a pulse such that $I_r\to 0$ as $\tau\to\infty$. For $|\mu|>1$ however the reflected intensity is periodically emitted since ξ varies periodically. Notice that for the case $|\mu|<1$, the maximum value of I_r is reached at ξ^* since during the evolution of the system $-\infty<\xi<\xi^*$. As opposed to pure superradiance where $|\mu|=0$, here the delay time τ^* depends on the initial azimuthal angle on the Bloch sphere and on the PCM gain coefficient $|\mu|$.

3.2. Nonzero external driving field case

Now let us consider the case when the atomic layer is coherently driven by an external field at exact resonance. The initial conditions are such that $D(0)=-1$, $X(0)=0$ and $Y(0)=0$.

Here the polarization components are given by

$$X=-\Omega_0\sin(\varphi-\tfrac{1}{2}\psi)\,[1-\exp(\Theta_x\xi)]\ ,\quad Y=\Omega_0\cos(\varphi-\tfrac{1}{2}\psi)\,[1-\exp(\Theta_y\xi)]\ . \tag{19}$$

And from eq. (14) we obtain

$$\tfrac{1}{2}(\mathrm{d}\xi/\mathrm{d}\tau)^2+\tfrac{1}{2}\Omega_0^2\{\sin^2(\varphi-\tfrac{1}{2}\psi)\ [1-\exp(\Theta_x\xi)]^2+\cos^2(\varphi-\tfrac{1}{2}\psi)\ [1-\exp(\Theta_y\xi)]^2\}=\tfrac{1}{2}\,. \tag{20}$$

Now let us analyze the dynamics for different values of μ and Ω_0.

In the case when $|\mu|<1$ and when $\Omega_0<1$ the particle performs an infinite motion $\xi\to-\infty$ as $\tau\to\infty$ and approaches an asymptotic Bloch vector:

$$X=-\Omega_0\sin(\varphi-\tfrac{1}{2}\psi)\,,\quad Y=\Omega_0\cos(\varphi-\tfrac{1}{2}\psi)\,,\quad D=-\sqrt{1-\Omega_0^2}\,.$$

When $\Omega_0=1$ the Bloch vector asymptotically approaches $X=0$, $Y=1$ and $D=0$. $\Omega_0=1$ is a bifurcation value since for $\Omega_0>1$ all the motions in eq. (20) become librational and therefore the Bloch vector describes periodical motions [19].

When $|\mu|=1$, one has $X=0$ for all values of $|\Omega_0\cos(\varphi-\tfrac{1}{2}\psi)|\leqslant 1$, the Bloch vector asymptotically approaches $X=0$, $Y=\Omega_0\cos(\varphi-\tfrac{1}{2}\psi)$ and $D=-\sqrt{1-\Omega_0^2\cos^2(\varphi-\tfrac{1}{2}\psi)}$. Notice that the population inversion depends on the PCM phase ψ relative to that of the external driving field φ. For $|\Omega_0\cos(\varphi-\tfrac{1}{2}\psi)|>1$ all the motions are librational in eq. (20) and the atomic system becomes fully excited and deexcited periodically.

Finally when $|\mu|>1$ with the exception of $\varphi-\tfrac{1}{2}\psi=0$, $\tfrac{1}{2}\pi$, π and $\tfrac{3}{2}\pi$, all the motions are librational in eq. (20). For $\varphi-\tfrac{1}{2}\psi=\tfrac{1}{2}\pi$, $\tfrac{3}{2}\pi$, provided $\Omega_0<1$, the atomic system asymptotically approaches $X=-\Omega_0$, $Y=0$ and $D=+\sqrt{1-\Omega_0^2}$. Notice that here the population inversion approaches an state in the upper Bloch hemisphere. While for $\varphi-\tfrac{1}{2}\psi=0$, π provided $\Omega_0<1$ the asymptotic steady state is $X=0$, $Y=\Omega_0$ and $D=-\sqrt{1-\Omega_0^2}$. For $\Omega_0>1$ all the motions on the Bloch sphere are periodical [19].

Let us write an expression for the intensity of the reflected field I_r. Using eqs. (12) and (19) we obtain

$$I_r=I_0\sin^2(\varphi-\tfrac{1}{2}\psi)\ \{\Theta_x[1-\exp(\Theta_x\xi)]+|\mu|\}^2+I_0\cos^2(\varphi-\tfrac{1}{2}\psi)\ \{\Theta_y[1-\exp(\Theta_y\xi)]-|\mu|\}^2\,, \tag{21}$$

where $I_0=(c/4\pi)E_0E_0^*$. Notice that in the ground or in the fully excited state $\xi=0$, and the reflected field will be given by $I_r=I_0|\mu|^2$. We point out that I_r depends in general on the relative phase between φ and $\tfrac{1}{2}\psi$ not only through ξ, as it may be seen in eq. (21). When the atomic system reaches coherently the steady state i.e. $\xi\to-\infty$ for $|\mu|<1$, $\Omega_0<1$, the reflected electric field $E_r=-E_0$ (perfect reflection), it means that at the thin layer position the total electric field $E^{(+)}=0$, as a result of what, in this case, $\mathrm{d}D/\mathrm{d}\tau=\mathrm{d}\alpha/\mathrm{d}\tau=0$, as one may see from eq. (7).

4. Steady-state phenomena (inclusion of incoherent damping)

In this section we study those collective effects at steady state that take place in our system when incoherent damping must be considered. In our model we assume that the PCM induced decay is irrelevant (see appendix). In what follows we will see that eqs. (10) show different qualitative behaviours in our system for $|\mu|>1+\Gamma_\perp$ and $|\mu|<1+\Gamma_\perp$. Our scheme of analysis in every case that we study is the following. First, we find the possible steady-state solutions, obtaining an equation of state. Secondly, we analyze the stability of these solutions discussing the evolution through the stable steady states under an adiabatic change of the external driving field or the PCM coefficient μ.

The stability or instability of the solutions may be established by studying the behaviour of the small perturbations of the system from the stationary state [25]. The following set of linearized equations for the small perturbations δX, δY and δD are obtained in general form from eqs. (10):

$$\mathrm{d}\delta X/\mathrm{d}\tau=(-\Gamma_\perp+\Theta_xD)\,\delta X+(\Theta_xX+\Omega_x)\,\delta D\,,$$

$$\mathrm{d}\delta Y/\mathrm{d}\tau=(-\Gamma_\perp+\Theta_yD)\,\delta Y+(\Theta_yY-\Omega_y)\,\delta D\,,$$

$$\mathrm{d}\delta D/\mathrm{d}\tau=-\Gamma_\parallel\,\delta D+(-2\Theta_xX-\Omega_x)\,\delta X+(-2\Theta_yY+\Omega_y)\,\delta Y\,. \tag{22}$$

According to the general procedure in linear stability analysis, one must look for solutions of the type

$$(\delta X, \delta Y, \delta D) = (\delta X_0, \delta Y_0, \delta D_0)\exp(\lambda t)\,.$$

The system of eqs. (22) has nontrivial solutions provided λ is an eigenvalue of the matrix A defined as

$$A=\begin{pmatrix} -\Gamma_\perp+\Theta_x D & 0 & \Theta_x X+\Omega_x \\ 0 & -\Gamma_\perp+\vartheta_y D & \Theta_y Y-\Omega_y \\ -2\Theta_x X-\Omega_x & -2\Theta_y Y+\Omega_y & -\Gamma_\parallel \end{pmatrix}.$$

In other words λ must satisfy the characteristic equation

$$\det(A-\lambda E)=0\,, \tag{23}$$

where E is the unity matrix. The particular characteristic equation in every analyzed case gives us the eigenvalues λ. The stationary solutions are stable if and only if the eigenvalues λ have a negative real part [25].

4.1. Zero external driving field case

Let us consider the case when there is no external field driving our system, i.e. $\Omega_0=\Omega_x=\Omega_y=0$. In this limit the system approaches always a steady state due to the incoherent damping processes. For $0<|\mu|<1+\Gamma_\perp$ the steady state is the ground state of the system, i.e. $X=Y=0$ and $D=-1$. For $|\mu|=1+\Gamma_\perp$, critical slowing down takes place in the system. In other words the system approaches the ground state with an "infinitely" slow damping. This point will be checked below.

As soon as $|\mu|>1+\Gamma_\perp$, the population inversion increases and the Bloch vector components are given by

$$X=\pm\sqrt{-\Gamma_\parallel(\Gamma_\perp+\Theta_x)}/|\Theta_x\,, \quad Y=0\,, \quad D=\Gamma_\perp/\Theta_x\,. \tag{24}$$

As $|\mu|\to\infty$, D steadily increases from -1 to 0 while X approaches zero. The component X is positive or negative according to the initial value of $X(0)$. X keeps the same sign of $X(0)$ at any time of the evolution. It is easy to prove it by noticing that eq. (10) for X in this case may be written as follows:

$$X=X(0)\lim_{\tau\to\infty}\exp\left(-\Gamma_\perp\tau+\vartheta_x\int_0^\tau \mathrm{d}q\, D(q)\right).$$

The X value in eq. (24) has been obtained by solving eq. (10) for D at steady state with $D=\Gamma_\perp/\Theta_x$ and $Y=0$. Now let us prove the stability conditions. For $|\mu|<1+\Gamma_\perp$, by inspection one may check that the ground state is stable. While for $|\mu|\geqslant 1+\Gamma_\perp$ we get from the general characteristic equation (23):

$$(-\Gamma_\perp+\Theta_y D-\lambda)\,[\lambda(\Gamma_\parallel+\lambda)+2\Theta_x^2X^2]=0\,. \tag{25}$$

The following eigenvalues are obtained:

$$\lambda_0=\Gamma_\perp(-1+\Theta_y/\Theta_x)<0\,, \quad \lambda_\pm=-\tfrac{1}{2}\Gamma_\parallel\pm\tfrac{1}{2}\sqrt{\Gamma_\parallel^2+8\Gamma_\parallel(\Gamma_\perp+\Theta_x)}\,. \tag{26}$$

$\mathrm{Re}(\lambda_\pm)<0$, since $\Theta_x=1-|\mu|<-\Gamma_\perp$ for this case. Critical slowing down takes place when $|\mu|=1+\Gamma_\perp$ since here $\lambda_+=0$. Notice that in this case as $|\mu|$ is increased the atomic system approaches the steady state describing a node in the X–D plane for $1+\Gamma_x<|\mu|\leqslant 1+\Gamma_\perp+\frac{1}{8}\Gamma_\parallel$ and a focus for $|\mu|>1+\Gamma_\perp+\frac{1}{8}\Gamma_\parallel$. From eq. (12) we get an expression for the reflected intensity when $|\mu|>1+\Gamma_\perp$ (when $|\mu|<1+\Gamma_\perp$, $I_r=0$):

$$I_r=(c/4\pi)\Gamma_\parallel B^2(|\mu|-1-\Gamma_\perp)\,. \tag{27}$$

In other words the reflected intensity is linearly proportional to the PCM gain coefficient $|\mu|$, as opposed to the case when an external coherent field impinges upon the PCM where the reflected intensity is proportional to $|\mu|^2$. Notice that in eq. (24) the X polarization component is different from zero, contrary to the case of a single two-level atom in front of a PCM, where the dipole expectation value is zero [14,15].

Next we will study the effects induced by an external coherent field on the dynamics of our atomic thin film in the presence of a PCM. We will concentrate on two particular cases that show new effects. These particular cases are $\varphi-\frac{1}{2}\psi=0$ and $\varphi-\frac{1}{2}\psi=\pi/2$.

4.2. Zero phase case

Let us consider the zero phase case $\varphi-\frac{1}{2}\psi=0$ i.e. $\Omega_x=0$ for different values of $|\mu|$.

(A) When $|\mu|<1+\Gamma_\perp$ from eq. (10) the following steady-state solutions for the polarization components are obtained:

$$X=0\,,\quad Y=\Omega_y D/(-\Gamma_\perp+\Theta_y D)\,. \tag{28}$$

The equation of state is obtained after inserting eq. (28) into eq. (10) for the population inversion when $\mathrm{d}D/\mathrm{d}\tau=0$:

$$\Omega^2=d(d-m)^2/(1-d)\,. \tag{29}$$

Here

$$\Omega^2=\Gamma_\perp\Omega_0^2/\Gamma_\parallel\,,\quad m=1+\Gamma_\perp/\Theta_y\,,\quad d=1+D\quad(0<d<1)\,.$$

For definiteness we assume that $\Omega\geqslant 0$.

In what follows we will use the population inversion definitions D or d. In the plane Ω^2-d, the critical points in the equation of state (29) are determined by

$$\partial\Omega^2/\partial d=0\,.$$

From eq. (29) we obtain:

$$\partial\Omega^2/\partial d=(d-m)(3d-m-2d^2)/(1-d)^2\,. \tag{30}$$

Since $0<|\mu|<1+\Gamma_\perp$ one has

$$1<(2+2\Gamma_\perp)/(2+\Gamma_\perp)<m<1+\Gamma_\perp$$

and therefore $\partial\Omega^2/\partial d=0$ at

$$d_\pm=\tfrac{3}{4}\left(1\pm\sqrt{1-\tfrac{8}{9}m}\right)\,. \tag{31}$$

Assuming that $\Gamma_\perp\ll 1$, $d_\pm$ may be approximated by

$$d_+\approx 1-\Gamma_\perp/\Theta_y\,,\quad d_-\approx\tfrac{1}{2}+\Gamma_\perp/\Theta_y\,. \tag{32}$$

Otherwise, if $9-8m<0$ there are no critical points at all and the system may be excited continuously as Ω^2 is increased. Now let us find the stability condition for the equation of state (29).

The characteristic equation (23) in this case takes the form

$$(-\Gamma_\perp+\Theta_x D-\lambda)\,[\lambda^2+(\Gamma_\parallel+\Gamma_\perp-\Theta_y D)\lambda+P/(\Gamma_\perp-\Theta_y D)^2]=0\,, \tag{33}$$

where

$$P=\Gamma_\parallel(\Gamma_\perp-\Theta_y D)^3+\Omega_0^2\Theta_y^2\Gamma_\perp(\Gamma_\perp+\Theta_x D)\,.$$

Therefore the eigenvalue

$$\lambda_0=-\Gamma_\perp+\Theta_x D<0\,.$$

Since $-1<D<0$ and $|\mu|>1+\Gamma_\perp$. On the other hand the real part of the two remaining eigenvalues λ may be analyzed as follows. Using the right-hand side (RHS) of the equation of state (29) instead of Ω_0^2 in the expression for P, we get

$$P=\frac{\Gamma_\parallel(d-m)^2}{\Theta_y^3}\frac{2d^2+m-3d}{1-d}. \tag{34}$$

Comparing eqs. (30) and (34) we find that $\partial\Omega^2/\partial d>0$ (<0) when $P>0$ (<0) since $m>1$. From algebra we know that the second factor in eq. (33), which is a quadratic equation in λ, has two roots λ_+ and λ_- with negative real parts if and only if $\lambda_++\lambda_-<0$ and $\lambda_+\lambda_->0$. $\lambda_++\lambda_-=-(\Gamma_\perp+\Gamma_\parallel-\Theta_yD)<0$ and $\lambda_+\lambda_-=P(\Gamma_\perp-\Theta_yD)^{-2}$. As a result of the last equality the stability condition in this case states that $\partial\Omega^2/\partial d>0$ must take place.

(B) For $|\mu|\geqslant 1+\Gamma_\perp$, to our knowledge a new kind of bistability occurs here. When the external field is larger than a critical value $\Omega>\Omega_c$ the equation of state is that given by eq. (29) with the difference that $|\mu|\geqslant 1+\Gamma_\perp$. While for $\Omega<\Omega_c$ the atomic system follows another stable-state equation, where the population inversion remains constant. As Ω is increased from a zero value the system passes from the latter to the former state by showing instabilities. Let us prove this behaviour.

When $\Omega<\Omega_c$ (Ω_c will be determined below) the steady state of the system is given by

$$X=\pm\sqrt{-\Gamma_\parallel/\Theta_x}\,\sqrt{1+\Gamma_\perp/\Theta_x+\Omega^2\Theta_x\Theta_y^2/\Gamma_\perp(\Theta_y-\Theta_x)^2},\quad Y=\Omega_0/(1-\Theta_x/\Theta_y),\quad D=\Gamma_\perp/\Theta_x. \tag{35}$$

Eq. (35) was similarly obtained as eq. (24). As expected eqs. (24) and (35) coincide when there is no external field, i.e. $\Omega=0$. Notice that as Ω increases, the population inversion D remains constant, Y increases while the absolute value of X decreases continuously. Again, as in the previous case X at steady state has the sign of $X(0)$. Once $\Omega=\Omega_c$, the polarization component $X=0$ where

$$\Omega_c^2=d_c(d_c-m)^2/(1-d_c),\quad d_c=1+\Gamma_\perp/\Theta_x$$

and at this moment the state described by eq. (35) is no longer stable as $\Omega\geqslant\Omega_c$. Any further adiabatical increase of Ω makes the system to have an evolution through those states given by eqs. (28) and (29) with the only difference that this time $|\mu|>1+\Gamma_\perp$. Let us prove the stability of states given by eq. (35).

The characteristic equation (23) in this case takes the form

$$\lambda^3+(\Gamma_\parallel+\Gamma_\perp-\Theta_yD)\lambda^2+[\Gamma_\parallel(\Gamma_\perp-\Theta_yD)+2\Theta_x^2X^2+(\Theta_yY-\Omega_y)(2\Theta_yY-\Omega_y)]\lambda-2\Theta_x^2X^2(-\Gamma_\perp+\Theta_yD)=0, \tag{36}$$

where the Bloch vector components are given by eq. (35). We must prove that the roots of eq. (36) have negative real parts. It is well known in algebra [26] that in the case of a cubic equation

$$a_0\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0. \tag{37}$$

The Hurwitz theorem states that eq. (37) has zeros with negative real parts if the following conditions are fulfilled:

$$T_0=a_0>0,\quad T_1=a_1>0,\quad T_2=a_1a_2-a_3a_0>0,\quad T_3=a_3T_2>0. \tag{38}$$

Comparing eqs. (36) and (37) one finds that $T_0=1>0$,

$$T_1=\Gamma_\parallel+\Gamma_\perp-\Theta_yD>0$$

and that

$$a_3=-2\Theta_x^2X^2(-\Gamma_\perp+\Theta_yD)>0.$$

Therefore if we prove that $T_2>0$, this regime will be stable. Using eqs. (35), (36) and (38), we find an expression for T_2:

$$T_2=-\frac{\Omega^2\Theta_y^2\Gamma_\parallel}{(\Theta_y-\Theta_x)^2\Gamma_\perp}\{2\Theta_x^2\Gamma_\parallel-[\Gamma_\parallel\Theta_x+\Gamma_\perp(\Theta_x-\Theta_y)](\Theta_y+\Theta_x)\}$$

$$-2\Gamma_\parallel^2(\Theta_x+\Gamma_\perp)+\frac{\Gamma_\perp\Gamma_\parallel(\Theta_x-\Theta_y)}{\Theta_x^2}[\Gamma_\parallel\Theta_x+\Gamma_\perp(\Theta_x-\Theta_y)]\,. \quad (39)$$

It is easy to check that when $\Omega=0$, i.e. there is no external field, $T_2>0$. Since T_2 is a decreasing function depending linearly on Ω^2, its minimum value will occur at $\Omega=\Omega_c$ in the interval $0<\Omega<\Omega_c$. We get

$$T_2(\Omega=\Omega_c)=-(2/\Theta_x)\Gamma_\parallel\Gamma_\perp(1+\Gamma_\perp)(\Gamma_\parallel+\Gamma_\perp-\Gamma_\perp\Theta_y/\Theta_x)>0\,. \quad (40)$$

In this way we have shown that for all the possible Ω values in this regime, $T_2>0$. As it was already pointed out, at $\Omega=\Omega_c$ we have $X=0$ and therefore one of the roots in eq. (36) is zero and as a result the system becomes unstable. In other words, in the plane Ω^2–d the regime given by eq. (35) is unstable, precisely at the point of intersection of the line $d=d_c=1+\Gamma_\perp/\Theta_x$ with the equation of state (29) when $|\mu|>1+\Gamma_\perp$. For $\Omega>\Omega_c$ the state equation (29) now becomes stable since for $d>d_c$, $\lambda_0=-\Gamma_\perp+\Theta_x D$ becomes negative. As in the previous case $\partial\Omega^2/\partial d>0$ must be fulfilled.

As a result one has that if line $d=d_c$ in the Ω^2–d plane intersects equation of state (29) at $d<d_-$, then there is a continuous transition between states described by eqs. (29) and (35). While if the intersection takes place in the region where $\partial^2\Omega/\partial d<0$ in eq. (29), the system may jump only to the upper branch of eq. (29) ($d>d_+$) to the corresponding value d. The system cannot jump to the lower branch ($d<d_-$) for a given Ω_c since there it is unstable: $\lambda_0>0$ for $d<d_c$. By comparing eq. (32) for d_+ and d_- one has that always $d_+>d_c$ while d_c may be larger or smaller than d_- according to the $|\mu|$ values.

In conclusion, it has been shown that the whole discussion given for the case $|\mu|<1+\Gamma_\perp$ still holds when $|\mu|>1+\Gamma_\perp$ provided $\Omega>\Omega_c$ and therefore $d>d_c$, since in the proof for the stability of the steady state given by eqs. (28) and (29), the conditions for $\mathrm{Re}(\lambda_\pm)<0$ were independent of the $|\mu|$ values. If one is interested in the intensity of the reflected field at $|\mu|>1+\Gamma_\perp$, one must use the polarization components X and Y given by eq. (35) in eq. (12) for $\Omega<\Omega_c$ and eqs. (28) for $\Omega>\Omega_c$.

4.3. $\frac{1}{2}\pi$ phase case

Let us consider the $\frac{1}{2}\pi$ phase case: $\varphi-\frac{1}{2}\psi=\frac{1}{2}\pi$ i.e. $\Omega_y=0$ for different $|\mu|$ values. From eq. (10) the following steady-state solutions are obtained for the polarization components (for all $|\mu|$ values):

$$X=-\Omega_x D/(-\Gamma_\perp+\Theta_x D)\,,\quad Y=0\,. \quad (41)$$

The state equation is given as in eq. (29) by

$$\Omega^2=d(d-m)^2/(1-d)\,, \quad (42)$$

where the notations are the same as in eq. (29) with the exception that here

$$m=1+\Gamma_\perp/\Theta_x\,. \quad (43)$$

Here as in the previous case we may classify the behaviour of the system according to the $|\mu|$ values. For $0<|\mu|<1+\Gamma_\perp$ bistability may occur while for $|\mu|>1+\Gamma_\perp$ the system may get excited or deexcited starting from the state given by eq. (24) as the intensity of the external field is increased. As the system deexcites it may become unstable for a certain value of the external field.

Taking into account that $\Omega\geqslant 0$, another equivalent expression for X may be obtained by inserting equation of state (42) in eq. (41):

$$X=(d-m)/|d-m|)\sqrt{\Gamma_\parallel/\Gamma_\perp}\sqrt{d(1-d)}\,. \quad (44)$$

For $d-m=0$ one has that $\Omega=0$. In the limit as d approaches m from the right or from the left X is positive or negative respectively. With this remark it is easy to check that eq. (44) coincides with the corresponding values in eq. (24) for $\Omega=0$. When $|\mu|<1+\Gamma_\perp$ here $d=0$ and $X=0$ and when $|\mu|>1+\Gamma_\perp$, one has $d\to m$.

As in the previous subsection, condition $\partial\Omega^2/\partial d=0$ determines the critical points. Notice that eqs. (30) and (31) are valid in the present case provided m is defined by eq. (43). Regarding the stability conditions, from the general characteristic equation (23) we obtain for all $|\mu|$ values:

$$(-\Gamma_\perp+\Theta_y D-\lambda)[\lambda^2+(\Gamma_\perp+\Gamma_\parallel-\Theta_x D)\lambda+Q]=0\,, \tag{45}$$

where

$$Q=2\Theta_x^2X^2+3\Theta_x\Omega_xX+\Omega_x^2+\Gamma_\parallel(\Gamma_\perp-\Theta_xD)\,.$$

Q may be rewritten using the equation of state (42). We obtain

$$Q=\Theta_x\Gamma_\parallel(2d^2+m-3d)/(1-d)=2\Gamma_\parallel\Theta_x(d-d_+)(d-d_-)/(1-d)\,,$$

where $d_\pm$ are given by formula (31). In eq. (45) we observe that the root

$$\lambda_0=-\Gamma_\perp+\Theta_yD<0\,.$$

Therefore one must concentrate on the second factor of eq. (45) enclosed in square brackets. This is a quadratic equation in λ with roots λ_+ and λ_-. Next we analyze the stability for the $0<|\mu|<1+\Gamma_\perp$ and $|\mu|>1+\Gamma_\perp$ cases.

When $0<|\mu|<1+\Gamma_\perp$, we have that for any Ω value roots $\lambda_\pm$ are such that

$$\lambda_++\lambda_-=-(\Gamma_\parallel+\Gamma_\perp-\Theta_xD)<-\Gamma_\parallel<0\,.$$

This upper bound for $\lambda_-+\lambda_+$ has been obtained by setting $|\mu|=1+\Gamma_\perp$ and $D=-1$. Both roots $\lambda_\pm$ have negative real parts if in addition $\lambda_+\lambda_->0$, i.e. $Q>0$ is fulfilled. We will show next that that is the case.

For $0<|\mu|<1-8\Gamma_\perp$ ($\Theta_x>0$) from eq. (43) one has that $1+\Gamma_\perp<m<\frac{9}{8}$. The critical values given by roots $d_\pm$ in eq. (31) are such that $0<d_-<d_+<1$. As a result $Q>0$ only for $0<d<d_-$ and $d_+<d<1$. In these intervals $\partial\Omega^2/\partial d>0$ as it follows from eq. (30).

For $1-8\Gamma_\perp<|\mu|<1$ ($\Theta_x>0$) one has that $\frac{9}{8}<m<\infty$. This time both roots $d_\pm$ are complex conjugates and $Q>0$ for any Ω value. Here all the stationary solutions are stable and the atomic system continuously gets excited as Ω is increased.

For $1<|\mu|<1+\Gamma_\perp$ ($\Theta_x<0$), we have $-\infty<m<0$. Roots $d_\pm$ are such that $d_-<0$ and $d_+>\frac{3}{2}$ and therefore the product $(d-d_+)(d-d_-)<0$. As a result $Q>0$ and all the stationary states are stable. For these $|\mu|$ values the system does not show bistability.

When $|\mu|>1+\Gamma_\perp$ one has $0<m<1$. Roots $\lambda_\pm$ must satisfy

$$\lambda_++\lambda_-=-(\Gamma_\perp+\Gamma_\parallel-\Theta_xD)<0\,.$$

This condition is fulfilled for $d>m^*=1+(\Gamma_\parallel+\Gamma_\perp)/\Theta_x$. On the other hand roots $d_\pm$ are such that $0<d_-<1$ and $d_+>1$. As a result since $\Theta_x<0$, we have that $Q>0$, provided $d>d_-$. From eq. (30) one has that $\partial\Omega^2/\partial d=0$ in the interval $0<d<1$ at $d=m$ and at $d=d_-$. Here we need to point out that always $m>m^*$ and that m^* may be larger or smaller than d_- depending on the $|\mu|$ values, as it may be checked. In short, states given by eqs. (41) and (42) are stable provided $d>\max(m^*, d_-)$ when $|\mu|>1+\Gamma_\perp$.

Now let us discuss the evolution of our system through the steady states as the external coherent field is adiabatically changed for $|\mu|>1+\Gamma_\perp$. Let us suppose that at the beginning there is no external field, in which case the state of the system is described by eq. (24). With a finite applied field ($\Omega\geqslant0$) the system may have two stable steady states with different energies, i.e. with different d values, according to the initial conditions. Assuming that the initial conditions are given by eq. (24) such that $X>0$ any further increase of the external field will draw the system through states that belong to the upper branch ($d>m$), in this way the steady state changes continuously since the characteristic equation gives $\mathrm{Re}(\lambda)<0$ for all λ at $d=m$. For $\Omega\to\infty$ the atomic system saturates: $d\to 1$ and $X\to 0$. If now we decide for a given state in the upper branch $(d>m)$ to decrease the external field, the inversion d diminishes until $d=m$ at $\Omega=0$. If now we assume that the initial state of the system corresponds to $X<0$ in eq. (24) any further increase of the external field will draw the atomic system through states with $d<m$, i.e. the inversion d diminishes $(\partial\Omega^2/\partial d<0)$. To our knowledge this is a new effect not reported before in the context of optical bistability in the bad cavity limit for a system of two-level atoms [3]. Provided $m^*>d_-$, this behaviour will take place until $\Omega=\Omega^*$ is reached at $d=m^*$. In this branch $\lambda_++\lambda_-<0$ for $\Omega<\Omega^*$ in $m^*<d<m$ and $\lambda_++\lambda_-\geqslant 0$ for $\Omega\geqslant\Omega^*$ in $d\leqslant m^*$, while $Q>0$ in the entire branch $d_-<d<m$. As a result roots $\lambda_\pm$ as $\Omega\to\Omega^*$ must be complex conjugated and at $d=m^*$, $\mathrm{Re}(\lambda_\pm)=0$. At this point the system will jump to the only stable steady state possible lying in the upper branch and corresponding to $\Omega=\Omega^*$, where the sign of X is positive and the reflected field I_r will be different, as we will see next. In the case when $d_->m^*$ the above mentioned dynamics will take place between $d_-<d<m$, since $\lambda_-+\lambda_+>0$ for $d<d_-$.

Let us obtain the expression for the reflected intensity in this case. Using equation of state (42) with the general expression for the reflected-field intensity (12) one finds

$$I_r=\frac{c}{4\pi}\frac{\Gamma_\parallel}{\Gamma_\perp}B^2\frac{d(d-m)^2}{1-d}\left((1-|\mu|)\frac{1-d}{d-m}-|\mu|\right)^2. \tag{46}$$

In particular, regarding the case $|\mu|>1+\Gamma_\perp$, one has that for a given value of Ω the reflected intensities in the upper branch and in the lower stable branch $(d<m)$ are in general different. It is easy to check that in the limit when $\Omega\to 0$ eq. (46) coincides with eq. (27).

5. Conclusions

In the present paper we studied the interaction between a phase conjugated mirror and a layer of two-level atoms with thickness much smaller than the emitted wavelength. We used the semi-classical, rotating-wave, dipole and slowly varying-amplitude approximations. The atomic layer is driven by an external field at exact resonance and the incoherent damping is supposed to be not affected by the PCM (see appendix). In this system we find new effects that are described below.

In the coherent case by establishing an analogy between the dynamics of our system and the motion of an imaginary particle in a potential well, we find out that in the absence of any external field and for different PCM gain coefficients the dynamics is similar to that of a point ensemble of two-level atoms interacting with a PCM. The point model was originally proposed and studied by Cook and Milonni [16] and was further analyzed by Pando [17]. Here if a coherent pulse is emitted in our atomic system, which we assume is initially nearly inverted, the time when the reflected intensity maximum appears will depend not only on the initial depletion angle but also on the initial azimuthal angle and on the PCM gain coefficient, as opposed to the case of pure superradiance.

When the external driving field is present and the initial conditions are given by the ground state of the system, one has that for $|\mu|<1$ and $\Omega_0<1$, where Ω_0 is a normalized Rabi frequency, the Bloch vector approaches asymptotically a steady state while for $\Omega_0>1$ the motions on the Bloch sphere are periodical. For $|\mu|=1$ and when $|\Omega_0\cos(\varphi-\frac{1}{2}\psi)|\leqslant 1$, where φ is the phase of the external electric field and ψ is the PCM intrinsic phase, the Bloch vector approaches a steady state. Here the population inversion will depend on the relative phase $\varphi-\frac{1}{2}\psi$. For $|\Omega_0\cos(\varphi-\frac{1}{2}\psi)|>1$ all the motions on the Bloch sphere are periodical.

For $|\mu|>1$ and for any Ω_0 value the Bloch vector describes periodical motions with the exception on the relative phases $\varphi-\frac{1}{2}\psi=0$, $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$. In particular when $\varphi-\frac{1}{2}\psi=\frac{1}{2}\pi$ the system approaches asymptotically a steady state with a population inversion in the upper Block hemisphere, i.e. the atomic system gets asymptotically inverted during the coherent interaction.

When the atomic layer reaches the steady state through incoherent damping, several bifurcation phenomena may arise in the atomic system by changing the external field or the PCM gain coefficient μ. In all the analyzed cases a qualitative change in the behaviour of the system takes place between $|\mu|<1+\Gamma_\perp$ and $|\mu|>1+\Gamma_\perp$ as the external field changes. Here $\Gamma_\perp$ is a renormalized transverse relaxation rate $\Gamma_\perp \ll 1$.

When there is no external driving field, we find that for $|\mu|<1+\Gamma_\perp$ the atomic system remains in its ground state, while for $|\mu|>1+\Gamma_\perp$, the atoms of the layer become excited, but with a nonzero polarization as opposed to a single two-level atom in front of a PCM, where the polarization expectation value is zero [14,15].

In the presence of an external driving field we study two cases according to $\varphi-\frac{1}{2}\psi=0$ or $\frac{1}{2}\pi$. The reason is that both show a different qualitative behaviour.

For $\varphi-\frac{1}{2}\psi=0$ we found out that for $|\mu|<1+\Gamma_\perp$ bistability may arise in the system as the Rabi frequency Ω is changed. Ω is a renormalized value of Ω_0. For $|\mu|>1+\Gamma_\perp$ a new effect takes place. As we increase Ω starting from a zero value, the population inversion does not change, it remains constant in spite of the fact that the polarization components change. It happens until a critical external field corresponding to Ω_c is reached. At this point the system becomes unstable and may pass continuously or discontinuously to a stable ready state described by an equation of state similar to that when $|\mu|<1+\Gamma_\perp$.

When $\varphi-\frac{1}{2}\psi=\frac{1}{2}\pi$ and for $|\mu|<1+\Gamma_\perp$, as the external field is changed the system shows bistability provided $|\mu|<1-8\Gamma_\perp$. On the other hand for $1-8\Gamma_\perp<|\mu|<1+\Gamma_\perp$ the system gets continuously excited as Ω increases. For $|\mu|>1+\Gamma_\perp$, as the Rabi frequency Ω is increased beginning from a zero value, the population inversion in the atomic system may continuously increase or decrease according to its steady state at $\Omega=0$, from which the atomic system starts its adiabatical evolution. If in the system its population inversion increases, it will get saturated as $\Omega\to\infty$. But if in the system its population inversion decreases, when a certain critical external field is reached corresponding to Ω^*, the system will jump to the only possible stable steady state for $\Omega=\Omega^*$ in the branch where the population inversion increases as the external field gets larger.

In all the studied cases a proof for the stability of the solutions has been given. In addition we found explicit expressions for the reflected field I_r. A justification of our model has been given in the appendix.

Acknowledgements

We are grateful for the hospitality of the Condensed Matter group at the International Center for Theoretical Physics.

Appendix

Here we like to discuss the manner in which the relaxation terms due to a nonradiative or radiative bath may appear in eq. (10). As for the radiative bath a discussion will be given at the end of the appendix. In section 2 we stated that the relaxation terms may be nonradiative in our model. When radiative relaxation is the only possible relaxation mechanism, one must start by looking at the relaxation of a two-level atom placed in front (in the radiation zone) of a PCM. Since the paper of Agarwal [12], which for the first time considered this problem, some work has been done recently [13–15,18]. In papers [14,15], in particular, it was found that the damping rates for the polarization components are

$$\gamma_\pm=\tfrac{1}{2}\gamma_0(1\pm 2\beta|\mu|+2\beta|\mu|^2)\,,$$

where γ_0 is the spontaneous emission rate in vacuum, β is the fraction of solid angle formed by the two-level atom and the boundary of the PCM. In our case $\beta=\frac{1}{2}$, $|\mu|$ is the PCM gain coefficient. $|\mu|$ is assumed not to change over the width of the spectrum emitted by the atomic dipole [14,15]. On the other hand it has been found [14,15] that the population of the upper state n_2 relaxes at a rate γ_z,

$$\gamma_z=\gamma_0(1+2\beta|\mu|^2)\,,$$

and its steady state is

$$n_2=\beta|\mu|^2/(1+2\beta|\mu|^2)\,.$$

Therefore for a point ensemble of two-level atoms in front of a PCM, in the simplest semi-classical description where dipole–dipole interactions are not taken into account, one must consider the macroscopic radiation-reaction field and the macroscopic PCM radiation field, acting on every two-level atom. Every two-level atom relaxes in the bath modified by the PCM. Keeping in mind the abovementioned results, here we are assuming that the two-level atoms are strongly coupled to a nonradiative bath that does not interact with the PCM. To this physical situation corresponds a system of impurity ions (atoms) localized inside a crystal or glass, where the bath is the phonon reservoir of the crystal or glass. In our case the impurities are formed by the thin film of two-level atoms. Another possible system is that formed by a fixed vibrational mode of a large molecule consisting of many atoms, interacting weakly with the remaining molecular modes, which form the bath.

In our model we are considering a situation where $\gamma_\parallel/\gamma_0$, $\gamma_\perp/\gamma_0\gg 1$, $|\mu|^2$. These inequalities do not contradict $\gamma_{\parallel,\perp}/g=\Gamma_{\perp,\parallel}\ll 1$, g being the cooperative relaxation rate. Therefore in the bath formed by the reservoir of nonradiative excitations and by the vacuum-field reservoir modified by the PCM [14,15] these inequalities entail that the polarization components X and Y relax at the same rate $\Gamma_\perp$ and that the upper-level population relaxes according to

$$\mathrm{d}n_2/\mathrm{d}t=-\gamma_z n_2+\gamma_0\beta|\mu|^2-\gamma_\parallel n_2$$

and therefore

$$n_2=\lim_{\gamma_\parallel/\gamma_0\to\infty}\frac{\beta|\mu|^2}{1+2\beta|\mu|^2+\gamma_\parallel/\gamma_0}=0\,.$$

As a result in this environment an atom is in its ground state at equilibrium. Here one must understand the limit $\gamma_\parallel/\gamma_0\to\infty$, in the sense of the abovementioned inequalities, which put an upper limit to the $|\mu|$ values. We believe that this model justifies the fact that the emitted photons appear as a result of induced transitions due to the coherent electric field at the atomic layer position.

Another system that may be described by equations similar to eqs. (10) is the following. Firstly, let us recall that if the incident electric field may be described as a plane-wave interacting with a dipole, the incident radiation and the stimulated emission will have the same phase, direction of propagation and frequency [27]. Due to the interference between the dipole field and the incident field the flow of energy where both fields constructively superimpose will take place effectively within the angle θ:

$$\theta=\sqrt{\lambda/z}\,,$$

where z is the distance from the dipole position [27] and λ is the emitted wavelength. For $z\gg\lambda$ constructive superposition of both fields will basically occur only near the z-axis. On the other hand for mathematical simplicity in most problems in quantum optics it is assumed that the field is a plane wave, that is, the models do not take into account the finite transverse extent of the field or the medium [27]. Our model falls within this approximation. A plane wave is better approximated by a gaussian beam, whose diffraction angle is given by $\theta^*=\lambda/\pi W_0$ [28], where W_0 is the minimum effective radius of the beam width (beam waist), which we assume is localized at the atomic sample position. If in the radiation zone ($z\gg\lambda$) the PCM is centered at the z-axis of propagation, where the PCM radius $r>W_0+\theta z=W_0+\sqrt{\lambda z}$ almost all the stimulated emission will reach the PCM front port. On the other hand all the external incident radiation reaches the PCM front port if its radius $r\approx W_0+\lambda z/\pi W_0$. Typical values at optical frequencies are $\lambda\approx 10^{-4}$ cm and $W_0\approx 10^{-2}$ cm. Therefore for a distance $z=1$ cm from the thin atomic sample we obtain $r\approx 1$ mm, that is a typical PCM radius [11]. Furthermore the solid angle $\beta\approx\pi r^2/4\pi z^2\approx 10^{-2}\ll 1$. Therefore the fraction of photons emitted by spontaneous radiation into the solid angle β is negligible and the PCM basically does not influence the decay rates, unless very high $|\mu|$ values are considered. In the microwave spectral region similar results may be obtained using Rydberg atoms. Here typical values are $\lambda \approx 1$ mm and $W_0 \approx 1$ cm [29]. If the PCM is localized at $z = 10$ cm, we obtain the following estimates: PCM radius $r \approx 1$ cm and $\beta \approx 10^{-2} \ll 1$. Here we point out that phase conjugation has been observed recently for the first time in the microwave region in a liquid suspension of elonged microparticles via four-wave mixing processes [30].

References

[1] A.V. Andreev, Sov. Phys. Usp. 33 (1990) 997.
[2] M. Gross and S. Haroche, Phys. Rep. 93 (1982) 301.
[3] L.A. Lugiato, in: Progress in optics XXI, ed. E. Wolf (North-Holland, Amsterdam, 1984) p. 89–216.
[4] V.M. Agranovich, V.I. Rupasov and V.Ya. Chernyak, JETP Lett. 33 (1981) 185.
[5] M.G. Benedict, V.A. Malyshev, E.D. Trifonov and A.I. Zaitsev, Phys. Rev. A 43 (1991) 3845, and references therein.
[6] C. Leonardi, F. Persico and G. Vetri, Riv. Nuovo Cimento 9 (1986) 1.
[7] Y. Ben-Aryeh, C.M. Bowden and J.C. Englund, Phys. Rev. A 34 (1986) 3917.
[8] A.M. Basharov, Sov. Phys. JETP 67 (1988) 1741.
[9] C.L. Pando L., J. Mod. Optics 37 (1990) 1175.
[10] Y. Ben-Aryeh and C.M. Bowden, J. Optics Soc. Am. B 8 (1991) 1168, and references therein.
[11] R.A. Fisher, ed., Optical phase conjugation (Academic Press, New York, 1983).
[12] G.S. Agarwal, Optics Comm. 42 (1982) 205.
[13] E.J. Bochove, Phys. Rev. Lett. 59 (1987) 2547.
[14] P.W. Milonni, E.J. Bochove and R.J. Cook, J. Optics Soc. Am. B 6 (1989) 1932.
[15] B.H.W. Hendriks and G. Nienhuis, Phys. Rev. A 40 (1989) 1892.
[16] R.J. Cook and P.W. Milonni, IEEE J. Quantum Electron. QE-24 (1988) 1383.
[17] C.L. Pando L., Optics Comm. 77 (1990) 157.
[18] H.F. Arnoldus and T.F. George, Phys. Rev. A 43 (1991) 3675.
[19] C.L. Pando L., J. Mod. Optics 38 (1991) 215.
[20] J.D. Jackson, Classical electrodynamics (Wiley, New York, 1962).
[21] H. Haken, Light, Vol. 2 (North-Holland, Amsterdam, 1985).
[22] V.I. Rupasov and V.I. Yudson, Sov. Phys. JETP 66 (1987) 282.
[23] M.G. Benedict and E.D. Trifonov, Phys. Rev. A 38 (1988) 2854, and references therein.
[24] R. Frieberg, S.R. Hartman and J.T. Manassah, Phys. Rev. A 39 (1989) 3444.
[25] V.I. Arnold, Ordinary differential equations (MIT Press, Cambridge, MA, 1973).
[26] G.A. Korn and T.M. Korn, Mathematical handbook (McGraw-Hill, New York, 1968).
[27] M. Sargent, M.O. Scully and W.E. Lamb, Laser physics (Addison-Wesley, Reading, MA, 1974).
[28] A. Yariv and P. Yeh, Optical waves in crystals (Wiley, New York, 1984).
[29] L. Moi, P. Goy, M. Gross, J.M. Raimond, C. Fabre and S. Haroche, Phys. Rev. A 27 (1983) 2043.
[30] R. Shih, H.R. Fetterman, W.W. Ho, R. McGraw, D. Rogovin and B. Bobbs, Phys. Rev. Lett. 65 (1990) 579.