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Hilda A. Cerdeira

Ramakrishna Ramaswamy

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Hilda A. Cerdeira **, Ramakrishna Ramaswamy ***
International Centre for Theoretical Physics, Trieste, Italy,

and

Amir O. Caldeira
Instituto de Fisica "Gleb Wataghin", Universidade Estadual de Campinas,
13081 Campinas, S.P., Brazil.

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** Permanent address: Instituto de Fisica, UNICAMP, C.P. 6165, 13081 Campinas, S.P., Brazil.
*** Permanent address: School of Physical Sciences, Jawaharlal Nehru University, New Delhi 100 067, India.
Present address: Department of Theoretical Studies, Institute for Molecular Sciences, Myodaiji, Okazaki 444, Japan.

ABSTRACT

We study a periodically kicked quantum oscillator system in contact with a heat bath. Using the Caldeira–Leggett approach, we solve for the kernel of the Wigner function at all temperatures. Previous results for dissipative quantum maps are recovered as special limits of low damping and slow kicks when the system effectively becomes one-dimensional. We then define the Lyapunov exponent for this quantum system by computing the expectation value for the coordinate variable, by taking the average along a semiclassical trajectory weighted by the Wigner function. In the semiclassical limit, the Lyapunov exponent scales as a positive exponent of Planck's constant.

It has been well established[1,2] that for classical systems the onset of chaos is defined when a positive Lyapunov exponent, which measures the local tendency of nearby trajectories to either expand or contract onto each other, can be associated to the motion, which in turn represents extreme sensitivity to initial conditions. It is not obvious what criteria needs to be evolved in order to similarly define a quantum chaos. It is already well understood that a broadband spectrum, decaying autocorrelation function and diffusive energy growth are necessary in order to have ergodicity and mixing[3]. In the search for chaos in driven quantum systems in the past few years has led to the conclusion that quantum chaos does not exist as a phenomenon that persists in time[4]. Quantum chaos was interpreted as the property which causes a quantum system to behave statistically, in the sense that expectation values approach those predicted by the rules of statistical mechanics[5]. The time taken by the system to reach equilibrium has been termed "break time" by Chirikov[6]. A related time scale, called recurrence time, for bounded quantum systems, has been studied by Hogg and Huberman[7]. These times are related to the ergodic wandering of the system on the energy shell following nonlinear perturbation.

For Hamiltonian systems some criteria have been set which allow to distinguish between two types of motion one going regular and the other chaotic in the semiclassical limit. For a quantal system whose corresponding classical motion is chaotic, the distribution of energy level spacings are those of the Gaussian orthogonal ensemble (or the Gaussian unitary ensemble) of random matrix theory depending whether the system preserves or not time reversal invariance[4]. The concept has been recently generalized by Grobe et al.[8] who seem to obtain that the distribution of smallest energy spacings display linear and cubic repulsion under the condition of classicaly regular and chaotic motion respectively (linear repulsion being a rigourous property of the Poissonian random process in the plane. Cubic repulsion is typical of the non Hermitian random matrices.

The attempt to define a quantal equivalent to "sensitivity to initial conditions" has been taken either as a Kolmogorov entropy in Hilbert space or as a quantum Reynolds number applied to evolving quantum systems[9,10]. Also Ikeda and Toda have also shown that the rate of growth of perimeters of contours of probability density may have some analogy to the classical Lyapunov exponent[11].

The recent work of Cerdeira, Furuya and Huberman[12] (CHF) and that of Caldeira, Cerdeira and Ramaswamy[13] presented a general approach to the calculation of the quantum *analog of the Lyapunov exponent, in the semiclassical limit.* They considered the case of a damped harmonic oscillator of frequency ω subjected to periodic forcing with period τ, previously considered by Graham and Tél[14]. The dissipation was considered through the coupling of the oscillator to a heat reservoir, in this case the Hamiltonian being:

$$H = H_0 + H_{int} \tag{1}$$

with

$$H_0 = \hbar\omega a^\dagger a - g\left(\sqrt{\frac{\hbar}{2\omega}}(a+a^\dagger)\right)\sum_{n=-\infty}^{\infty}\delta(t-n\tau) \tag{2}$$

where a ($a^\dagger$) is the destruction (creation) operator for the oscillator and g(x) is the pulsing potential, and

$$H_{int} = \alpha\sum_i (aR_i^\dagger + a^\dagger R_i) \tag{3}$$

where $R_i(R_i^\dagger)$ are the bath operators (refs.). The distribution of the reservoir modes, which couple to the oscillator through the coupling constant α, is assumed to be such as to give the phenomenological damping constant, γ.

The entire evolution of the system can be studied via the density matrix $\rho(t)$, which evolves as:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H, \hat{\rho}] \tag{4}$$

It is convenient to take advantage of the natural periodicity of the problem (through the forcing) to obtain a quantum map, as described by Berry et al.[15]. Note that at times $t=n\tau$, the forcing term dominates, therefore

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[g(x), \hat{\rho}] \tag{5}$$

and

$$\hat{\rho}(n\tau+0) = e^{\frac{i}{\hbar}g(\hat{x})}\,\hat{\rho}(n\tau-0)\,e^{-\frac{i}{\hbar}g(\hat{x})} \tag{6}$$

which relates the density matrix just before the kick to the density matrix just after. Since we are interested in the semiclassical limit we use the Wigner representation

$$W(x,p,t) = \int \frac{d\eta}{2\pi h}\, e^{-i\frac{p\eta}{\hbar}}\, \tilde{\rho}\left(x+\frac{\eta}{2}, x-\frac{\eta}{2}, t\right) \tag{7}$$

where

$$\tilde{\rho}(q,q';t) = \int d\vec{R}\, \langle q, \vec{R} \mid \hat{\rho}(t) \mid q', \vec{R} \rangle \tag{8}$$

We get for the Wigner function

$$W(x,p,n\tau+0) = \int \frac{d\eta}{2\pi h}\, e^{-i\frac{p\eta}{\hbar}}\, e^{\left\{\frac{i}{\hbar}\left[g\left(x+\frac{\eta}{2}\right)-g\left(x-\frac{\eta}{2}\right)\right]\right\}} \times \tilde{\rho}\left(x+\frac{\eta}{2}, x-\frac{\eta}{2}; n\tau-0\right). \tag{9}$$

In between pulses the problem is solved using standard methods. With the conditions of weak coupling ($\gamma \ll \omega$) and high temperatures ($k_B T \gg \hbar\gamma$) a master equation can be written for the density matrix leading to a Fokker Planck equation for the Wigner function of the damped oscillator between pulses[14,16]. A non-local map can be constructed which relates the Wigner function W_{n+1} after the (n+1)th kick to that after the nth pulse, W_n, by

$$W_{n+1}(x,y) = \int dx_n \int dy_n\, K(x,y;x_n,y_n)\, W_n(x_n,y_n)\ , \tag{10}$$

where $y = (-p + \omega f(x)/E\ )/\omega$, $f(x) = g'(x)$, $E = \exp(-\gamma\tau)$ and K is a kernel given by

$$K(x,y;x_n,y_n) = \int \frac{d\xi}{2\pi} \int \frac{d\eta}{2\pi}\, \omega\, e^{-i\xi(x-x^0_{n+1}(x_n,y_n))} \times e^{i\eta\omega(y-y^0_{n+1}(x_n,y_n;x))}\, e^{-\frac{\hbar Q}{2\omega}(\xi^2+\omega^2\eta^2)}\, e^{iG(x,\eta;\hbar)}. \tag{11}$$

where

$$G = \hbar^{-1}\left[g\left(x+\eta\frac{\hbar}{2}\right) - g\left(x-\eta\frac{\hbar}{2}\right) - \eta\hbar g'(x)\right]\ . \tag{12}$$

$$Q = \frac{1-E^2}{2} \coth\left(\frac{\hbar\omega}{2k_B T}\right) , \qquad (13)$$

and

$$x^0_{n+1}(x_n,y_n) = -Ey_n + f(x_n) \qquad (14a)$$

$$y^0_{n+1}(x_n,y_n;x) = y^0_{n+1}(x_n,y_n) . \qquad (14b)$$

Those equations were derived for the particular choice of τ such that $\omega\tau = 2\pi$ (k+1/4) with k integer. In the limit of $\hbar \to 0$, the quantum effects on the dissipative map appear effectively as classical noise. Or expressed in a different way, for finite h the function is not longer localized at the classical values: there is broadening due to dissipation and nonlinearity! Using this fact CFH proposed an algorithm to calculate the Lyapunov exponent for a quantum system, following a field theoretic method used by Shraiman, Wayne and Martin for classical 1-dimensional systems[17], i.e.: through the calculation of correlation functions as a path integration over a sequence determined by the noisy chaotic map. **The crucial difference for quantal systems is that an average of a functional should be calculated over the corresponding fuzzy classical map, weighted with the kernel of the Wigner function of the quantum problem.** If we consider the high dissipation limit of the map described by Eqs. (8) it reduces to a one dimensional map of the form $x_{m+1} = f(x_m) + \xi_m$, where the random variable ξ with mean zero and variance σ^2, represents additive noise. For this map the Lyapunov exponent is defined by:

$$\lambda = \lim_{M\to\infty} \frac{1}{M} \ln \{ f'(x_0)\, i <x_M \xi_0> \} , \qquad (15)$$

$$i<x_M\xi_0> = Z^{-1}\int [Dx]\,[D\xi]\; i x_M \xi_0 \times \exp\left[\sum_m \left\{ i\xi_m(x_{m+1}-f(x_m)) - \frac{1}{2}\sigma^2\xi_m^2 \right\} \right] . \qquad (16)$$

Now, using the definition of CHF the average of a functional is defined by:

$$<F[(x,y)]> = Z^{-1}\int dx_N \int dy_N\, F(x_N,y_N)$$
$$\int dx_{N-1}\int dy_{N-1} F(x_{N-1},y_{N-1})\, K(x_N,y_N;x_{N-1},y_{N-1})$$
$$\int dx_{N-2}\int dy_{N-2}\, F(x_{N-2},y_{N-2})\, K(x_{N-1},y_{N-1};x_{N-2},y_{N-2})$$
$$\ldots \int dx_1\int dy_1\, F(x_1,y_1)\, K(x_2,y_2;x_1,y_1) \qquad (17)$$
$$\int dx_0\int dy_0\, F(x_0,y_0)\, K(x_1,y_1;x_0,y_0)\, W_0(x_0,y_0) .$$

In the particular problem we are describing we are interested in the Lyapunov exponent in the x direction, therefore we can integrate over the {y} sequence and its conjugate {η}, and after taking the strong dissipation limit, we obtain

$$<F[\{x\}]> = Z^{-1}\int dx_N\, F(x_N)\, e^{iG(x_N,0;\hbar)}$$

$$\prod_{n=1}^{N-1}\int dx_n\, F(x_n)\int\frac{d\xi_n}{2\pi}\, e^{-i\xi_n(x_{n+1}-f(x_n))-\frac{\sigma^2\xi_n^2}{2}}\, e^{iG(x_n,0;\hbar)} \tag{18}$$

$$\int dx_0\, F(x_0)\int\frac{d\xi_0}{2\pi}\, e^{-i\xi_0(x_1-f(x_0))-\frac{\sigma^2\xi_0^2}{2}}\int dy_0\, W_0\,(x_0,y_0)$$

Keeping lowest order on $\hbar$, we can neglect the function G and complete the calculation thus obtaining Eq.(10), which coincides with Shraiman et al.'s result. Therefore we see that our definition of a Lyapunov exponent for a quantum system has the proper limit when $\hbar\to 0$. The noise terms are Gaussian, with zero mean and variance given by

$$< \xi_n\, \xi_m > = \sigma^2\delta_{n,m} = \frac{\hbar}{2\omega}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\delta_{n,m}\ . \tag{19}$$

Using the classical results, we predict that the Lyapunov exponent behaves as

$$\lambda \;\propto\; \sigma^{\theta} \tag{20}$$

thus satisfying a scaling law in the vicinity of the transition as $\hbar^{0.18}$. **From this we infer that for a given control parameter, for which the system is classically integrable the introduction of quantum effects will make it chaotic, contrary to well known results of Hamiltonian systems.**

At this point we should make an interlude for a few comments. The algorithm due to CFH was originally applied to a system with a bi-dimensional map which collapses into a 1-dimensional map under high dissipation. Later Graham pointed out that it is not necessary to have a dissipative

quantum system to exploit the results of classical noisy maps in order to calculate Lyapunov exponents[18]. He showed that a scaling behavior can be found for a class of quasi-stochastic maps of the form

$$z_{n+1} + b\, z_{n-1} = f\,(r,\, z_n\,) + \xi_n \tag{21}$$

with r being tha control parameter, and b a parameter which describes dissipation ($|b|=1$ meaning conservative and $|b|<<1$, dissipative), all of them with period doubling bifurcations, although with an unstable fixed point for $|b|=1$. **We want to call the attention to the fact that the CFH algorithm does not depend on the characteristic route to chaos.** It is only an accident of the model under consideration that may lead the reader to believe that high dissipation and period doubling are necessary conditions for the application of the algorithm. **It only requires the knowledge of the Wigner function of the quantum problem.** The kernel of the Wigner function fixes the path of integration and the ability to find the classical map is simply helpful in numerical calculations.

When we lift the restriction that the interaction with the bath be taken in the rotationg wave approximation, the kernel of the Wigner function in between pulses can be found without restrictions of strength in the coupling constant, neither temperature, using the influence functional calculated by Caldeira and Leggett[19]. In this case the kernel can be written:

$$K\,(x,p;x',p') = \int\frac{d\xi\, d\eta}{2\pi\hbar}\, e^{-i\left(\frac{p\eta}{\hbar}+\frac{p'\xi}{\hbar}\right)}\times e^{\left\{\frac{i}{\hbar}\left[\,q\left(x+\frac{\eta}{2}\right)-q\left(x-\frac{\eta}{2}\right)\right]\right\}}$$

$$\times J\left(\,x+\frac{\eta}{2},\, x-\frac{\eta}{2},\, n\tau-0;\, x'+\frac{\xi}{2},\, x'-\frac{\xi}{2},\, (n-1)\tau-0\right) \tag{22}$$

where $J(x,y,t;x',y',t')$ is the influence functional of the problem.

With a slight change of notation $(x',p') \rightarrow (x_n,p_n)$ and $(\xi/\hbar,\eta/\hbar) \rightarrow (\xi,\eta)$ the kernel can be written as

$$K(x,p;x_n,p_n) =$$

$$\hbar \int \frac{d\eta}{2\pi}\, d\xi \; e^{-i\{p-l(x)\eta+p_n\xi\}} \times e^{\frac{i}{\hbar}\left[q\left(x+\frac{\hbar\eta}{2}\right) - q\left(x-\frac{\hbar\eta}{2}\right) - \eta\hbar f(x)\right]}$$

$$\times J\left(x+\frac{\hbar\eta}{2},\, x-\frac{\hbar\eta}{2},\, n\tau-0;\, x_n+\frac{\hbar\xi}{2},\, x_n-\frac{\hbar\eta}{2},\, (n-1)\tau+0\right) \tag{23}$$

Here we are making the markovian assumption that $J(x,y,t;x',y',t')=J(x,y,t-t';x',y',0)$. This assumption, which is needed to keep the initial conditions for the validity of the kernel from Caldeira and Leggett imposes the condition $k_BT >> \hbar\gamma$, therefore making impossible to consider the low temperature limit in this problem. Although it remains valid for any coupling strength.

The explicit form of the kernel can be written as

$$K(x,y;x_n,y_n) = \int \frac{d\eta}{2\pi}\frac{d\xi}{2\pi}\, L(x,y;x_n,y_n;\xi,\eta) \tag{24}$$

with

$$L(x,y;x_n,y_n;\xi,\eta) =$$

$$e^{iG\left(x,\frac{\eta}{M\omega};\hbar\right)} \times e^{-i\xi\left(x-\frac{K+M\gamma}{N}x_n+\frac{M\omega}{N}y_n-\frac{f(x_n)}{N}-\frac{iB(\tau)\hbar}{NM\omega}\eta\right)}$$
$$\times e^{i\eta\left(y+\frac{K-M\gamma}{M\omega}x-\frac{L}{M\omega}x_n\right)} \times e^{-\hbar\left(\frac{A(\tau)}{(M\omega)^2}\eta^2+\frac{C(\tau)}{N^2}\xi^2\right)} \tag{25}$$

The functions K, L and N are given by:

$$K(t) = M\omega \cot \omega t$$

$$L(t) = \frac{M\omega\, e^{-\gamma t}}{\sin \omega t} \tag{26}$$

$$N(t) = \frac{M\omega\, e^{\gamma t}}{\sin \omega t}$$

while A, B and C, in the weak coupling limit are given by (complete expressions can be found in Ref. 19):

$$A(\tau) \equiv \frac{M\omega}{4}\, e^{-2\gamma\tau} \frac{\coth\left(\frac{\hbar\omega}{2kT}\right)}{4\,\omega^2+\gamma^2}$$
$$\times \{ 4\,\omega^2 + 2\, e^{2\gamma\tau}(2\,\omega^2+\gamma^2) - 8\,\omega^2 e^{\gamma\tau} \} \tag{27}$$

$$B(\tau) = -\,8\, e^{-\gamma\tau} \frac{M\,\omega^2\,\gamma}{(4\,\omega^2+\gamma^2)} \coth\left(\frac{\hbar\omega}{2kT}\right)\{ 1 - e^{2\gamma\tau} \} \tag{28}$$

$$C(\tau) = \frac{M\,\omega}{4\,(4\,\omega^2+\gamma^2)} \coth\left(\frac{\hbar\omega}{2kT}\right)$$
$$\times \{ 4\,\omega^2 e^{2\gamma\tau} - 8\,\omega^2 e^{\gamma\tau} + 4\,(\gamma^2+\omega^2)\} \tag{29}$$

The Lyapunov exponent for this kernel has the following expression:

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \ln \Big\{ -i \frac{L}{M\omega} < x_N \eta_0 > + i \frac{K+M\gamma}{N} < x_N \xi_0 > \tag{30}$$

$$+ \frac{i}{N} < x_N f'(x_0) \xi_0 > \Big\}$$

The expressions for the correlation functions can be calculated in the semiclassical limit, and we obtain, for instance:

$$< x_n \xi_0 > = -\frac{2i}{Z} \frac{N^2}{4\hbar C_0} \int dx_n\, dx_{n-1} \ldots dx_0\, dy_0 \quad x_n\, W_0 (x_0, y_0) \tag{31}$$

$$\times \left[\prod_{k=0}^{n-1} \frac{e^{-\frac{\alpha^2_{n+1,k}}{C'_k}}}{\sqrt{\pi C'_k}} \right] \times \Omega_{n,n-1}$$

where

$$C'_k = \frac{4\hbar C_k}{N^2} \tag{32}$$

such that

$$C'_m = A' + C' - \frac{B'^2}{4 C'_{m-1}} \qquad \text{for } m = 0,1,\ldots,n-2$$

$$C'_0 = C \tag{33}$$

$$C'_{n-1} = C - \frac{B^2}{4 C_{n-2}}$$

$$\Omega_{n,n-1} = \Omega_{n-1,n-2} + \left[\prod_{k=1}^{n-1} \left(-\frac{B}{2 C_k} \right) \right] \times \alpha_{n,n-1} \tag{34}$$

$$\Omega_{2,1} = \alpha_{1,0} - \frac{B}{2 C_1} \alpha_{2,1}$$

$$\alpha_{n,n-1} = x_n - \frac{K+M\gamma}{N} x_{n-1} + \frac{L}{M\omega} x_{n-2} - \frac{f(x_{n-1})}{N} - \frac{B}{2 C_{n-2}} \alpha_{n-1,n-2} \tag{35}$$

$$\alpha_{1,0} = x_1 - \frac{K+M\gamma}{N} x_0 + \frac{M\omega}{N} y_0 - \frac{f(x_0)}{N} .$$

With the above recursive scheme any correlation function can be evaluated. Extensive numerical calculations using the algorithm are under way.

Minimizing the classical action we can calculate the classical map, for which we find:

$$x_n = \frac{K+M\gamma}{N} x_{n-1} - \frac{f(x_{n-1})}{N} + \frac{M\omega}{N} y_{n-1} \tag{36}$$

$$y_n = -\frac{K-M\gamma}{M\omega} x_n + \frac{L}{M\omega} x_{n-1}$$

The structure of the map, as a function of the damping constant and the noise as well as the bifurcation diagram can be seen in the figures that follow. The effect of dissipation can be easily seen by comparing figures 1 and 3, while that of the noise by comparing Figs. 1 and 2, or 3 and 4. In Figs. 5 and 6 we show the effect of dissipation on the bifurcation diagram.

Here we have shown a formalism, as well as the explicit formulas to be evaluated to calculate the Lyapunov exponent for a quantum system. Other methods have appeared in the literature, but their relevance and connection need to be explored. Our method has the advantage that the time evolution of the Lyapunov exponent can be studied and in this way providing a well defined way to define the "break time" of the system. For this more extensive numerical calculations are needed. The method is also attractive insofar permits a definition independently of dissipation and time dependent properties of the system, with a wide range of applications.

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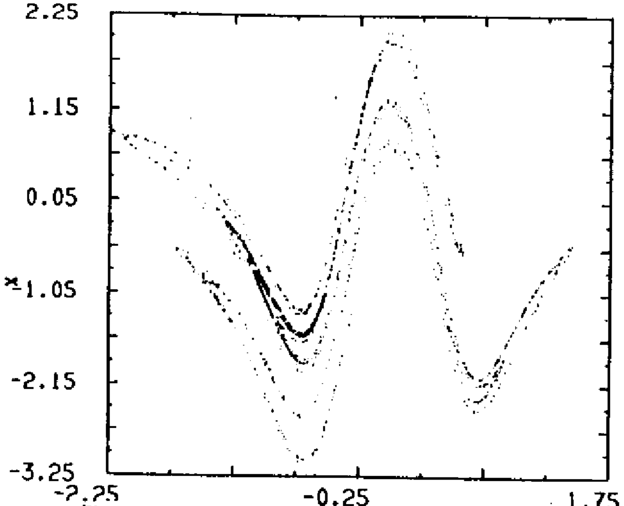

Fig.1_ Structure of the classical map in the x-y plane for the following parameters: $h=0.01$, $a=6.5$, $\gamma=0.0015$, $K=0$, $M=1$, $\omega=1$, $N=1.434$, $L=0.697$.

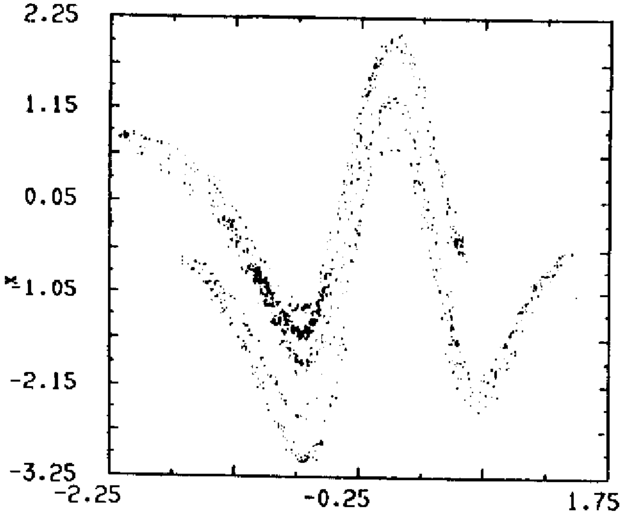

Fig.2_ Same as figure 1, with δ-correlated noise of variance $\sigma= 0.06$.

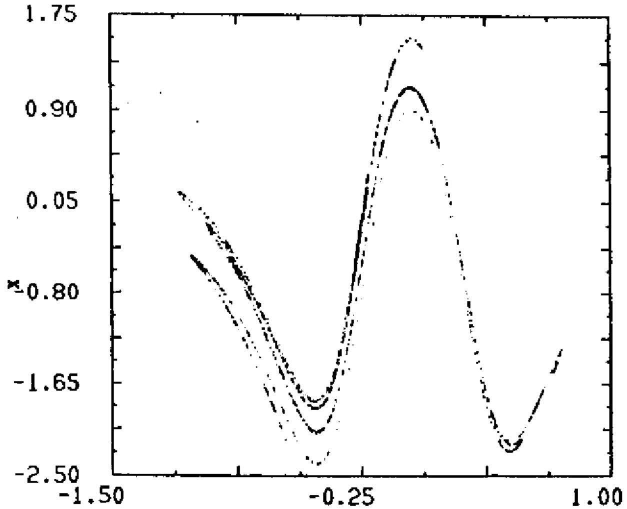

Fig.3_ Same as figure 1, with $a=6.5$, $\gamma=0.003$, $K=0$, $M=1$, $\omega=1$, $N=3.333$, $L=0.3$, without noise.

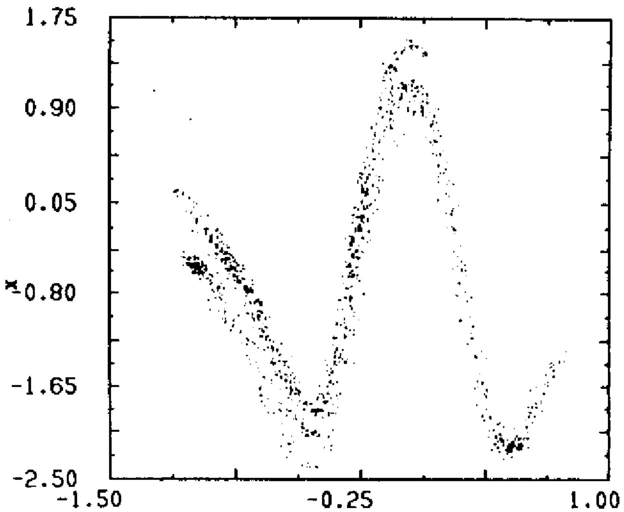

Fig.4_ Same as figure 2, with δ-correlated noise of variance $\sigma= 0.006$.

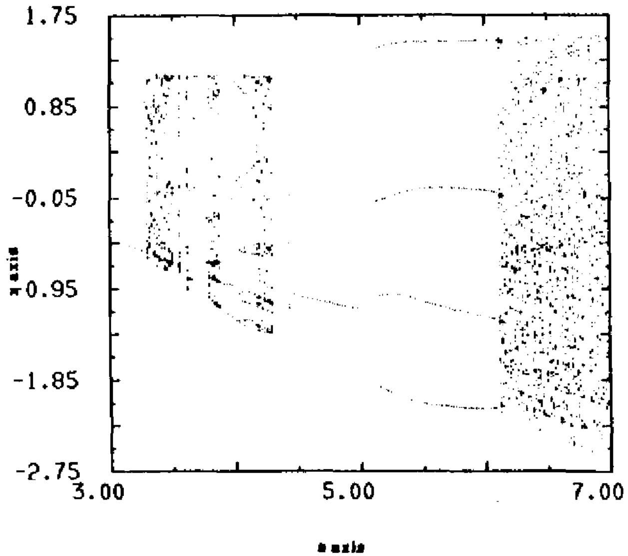

Fig.5_ Bifurcation diagram for γ=0.003.

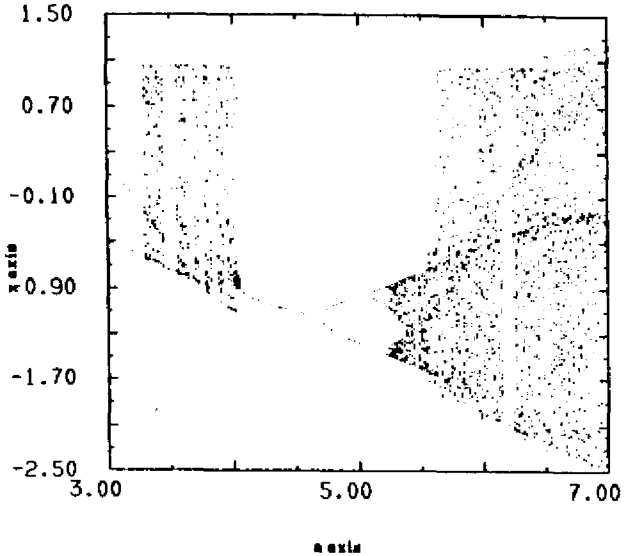

Fig.6_ Bifurcation diagram for γ=0.005.

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