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LONGITUDINAL ULTRASONIC ATTENUATION IN TYPE II SUPERCONDUCTORS: CLEAN LIMIT*

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Abstract—The attenuation of longitudinal sound in clean type II superconductors, near H_{c_2}, has been calculated using a non-perturbative method. It is shown that at low frequencies the attenuation rate depends strongly on the direction of propagation of the wave, and on impurity concentration.

1. INTRODUCTION

IN THIS paper we present a procedure for calculating the transport coefficients of clean type II superconductors near H_{c_2} which does not depend on an expansion in powers of the order parameter. The theory is used to study the attenuation of longitudinal sound in pure Niobium in magnetic fields close to the upper critical field. It is shown that the attenuation coefficient of longitudinal sound depends strongly on the direction of propagation, and is markedly dependent on impurity concentration even when $ql \gg 1$ and $\xi_0 \ll l$, here q is the wave vector of the sound wave, l is the electron mean free path and ξ_0 is the pure superconductor coherence distance.

2. THE ATTENUATION COEFFICIENT

In the low frequency limit $\omega_s < \pi T_{c0}$, provided $ql \gg 1$, the attenuation coefficient of longitudinal sound is given by

$$\alpha_L = Re \frac{q^2}{i\omega_s \rho_{\text{ion}} v_s}\left(\frac{P_F{}^2}{3m}\right)^2 \langle [n, n] \rangle (\mathbf{q}, \omega_s). \quad (1)$$

In equation (1) q and ω_s are the wave vector and frequency of the sound wave, p_F is the Fermi momentum and $\langle [n,n]\rangle(q, \omega_s)$ is the volume average of the density–density correlation function. The correlation function is obtained by analytic continuation of the thermal product $\langle [n, n] \rangle (q, \omega_0)$ from the set of discrete points $\omega_0 = 2m\pi T$ to $z = \omega_s - i\delta$. If we assume that, in the vicinity of H_{c_2}, the nearly uniform and constant magnetic field inside the superconductor can be replaced by its space average, the magnetic induction **B**, and treat the magnetic field dependence semiclassically, the thermal product can be written

$$\langle [n,n]\rangle (\mathbf{q}, \omega_0) = 2T \sum_{\omega} \int d^3r \int d^3r' \times e^{-i\mathbf{q}(\mathbf{r}-\mathbf{r}')}[G_{\omega+}(\mathbf{r},\mathbf{r}')G_{\omega}(\mathbf{r}',\mathbf{r}) - \int d^3r_1 \int d^3r_2 G^0_{-\omega_+}(\mathbf{r}-\mathbf{r}_1)G_{\omega_+}(\mathbf{r}',\mathbf{r}_1) \times V(\mathbf{r}_1,\mathbf{r}_2)G_{\omega}(\mathbf{r}_2,\mathbf{r}')G^0_{-\omega}(\mathbf{r}_2-\mathbf{r})] \quad (2)$$

here $\omega = (2n+1)\pi T$ and $\omega_+ = \omega + \omega_0$.
The Green's function $G_\omega(\mathbf{r}, \mathbf{r}')$ appearing in equation (2) satisfies the equation

$$G_\omega(\mathbf{r},\mathbf{r}') = G_\omega{}^0(\mathbf{r}-\mathbf{r}') - \int d^3r_1 \int d^3r_2 \times G_\omega{}^0(\mathbf{r}-\mathbf{r}_1)V(\mathbf{r}_1,\mathbf{r}_2)G^0_{-\omega}(\mathbf{r}_1-\mathbf{r}_2)G_\omega(\mathbf{r}_2,\mathbf{r}'), \quad (3)$$

$G^0(\mathbf{r}-\mathbf{r}')$ is the normal metal Green's function in the absence of a magnetic field and the function V is defined by:

$$V(\mathbf{r}_1, \mathbf{r}_2) = \Delta(\mathbf{r}_1)\Delta^*(\mathbf{r}_2)\, e^{-ieB(x_1+x_2)(y_1-y_2)} \quad (4)$$

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the magnetic field having been chosen to be in the z direction.

It has been shown[1] that iterative solutions to equation (3) lead in certain cases to unphysical results even in the limit $B \rightarrow H_{c_2}$; recently, however, Brandt *et al.*[2] have solved equation (3) approximately by a method which avoids the iteration procedure. These authors note that when $(H_{c_2}-B) \ll H_{c_2}$ the order parameter $\Delta(\mathbf{r})$ is given by the Abrikosov solution of the Ginzburg–Landau equations

$$\Delta(\mathbf{r}) = \sum_n C_n\, e^{in\alpha y}\, e^{-eB(x-\alpha n/2eB)^2} \tag{5}$$

and therefore $G(\mathbf{r}, \mathbf{r}')$ when considered as a function of sum and difference coordinates, has the periodicity of the flux line lattice with respect to the sum coordinate. Fourier analyzing equation (3) and noting that the dominant contribution to G is obtained if V is replaced by $\langle V\rangle$, an average over its sum coordinate they obtain in the infinite mean free path limit

$$G_\omega(\mathbf{p}, \mathbf{k}) = \delta_{k,0}\left[i\omega - \xi_p + \frac{i\sqrt{\pi}\Delta^2}{k_c V_F \sin\theta} W\left(\frac{i\omega + \xi_p}{K_c v_F \sin\theta}\right)\right]^{-1} \tag{6}$$

where the wave vector $k_c = (2eB)^{1/2}$ is inversely proportional to the spacing between flux lines,

$$\Delta^2 = |\overline{\Delta}|^2,$$

$$\xi_p = p^2/2m - \mu,$$

θ is the angle between $\mathbf{p}$ and $\mathbf{B}$ and

$$W(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, \frac{e^{-t^2}}{z-t}. \tag{7}$$

If we restrict our calculations to the clean limit $k_c l \gg 1$, scattering effects due to impurities are included in the theory by simply replacing ω by $\tilde{\omega} = \omega(1 + i/2\tau|\omega|)$. Corrections to the order parameter and the electromagnetic vertices are negligible to order $1/k_c l$ and $1/ql$ respectively. We should point out that as $k_c \rightarrow 0$ when $T \rightarrow T_c$ this theory is not valid too close to the transition temperature. Making this replacement and performing the analytic continuation of equation (2) we obtain

$$\alpha_L = \frac{-q}{v_s \rho_{\text{ion}}}\left(\frac{p_F^2}{3m}\right)^2 Im \int \frac{\mathrm{d}\omega}{2\pi i}[f(\omega+\omega_s) - f(\omega)]$$
$$\times [\langle[n, n]\rangle(\mathbf{q}; \omega+\omega_s + i/2\tau, \omega + i/2\tau)$$
$$-\langle[n; n]\rangle(\mathbf{q}; \omega+\omega_s + i/2\tau, \omega - i/2\tau)] \tag{8}$$

where

$$\langle[n, n]\rangle(\mathbf{q}; \omega+\omega_s + i/2\tau, \omega - i/2\tau)$$
$$= 2\int \frac{\mathrm{d}^3p}{(2\pi)^3} G_{\omega+\omega_s+i/2\tau}(\mathbf{p}+\mathbf{q}, 0)\, G_{\omega-i/2\tau}(\mathbf{p}, 0)$$
$$\times\left[1 - \int \frac{\mathrm{d}^3p'}{(2\pi)^3} G^0_{-(\omega+\omega_s+i/2\tau)}(\mathbf{p}+\mathbf{q}+\mathbf{p}')\right.$$
$$\left.\times G^0_{-(\omega-i/2\tau)}(\mathbf{p}+\mathbf{p}')v(\mathbf{p}', 0)\right] \tag{9}$$

$$V(\mathbf{p}', 0) = (\Delta/k_c)^2 2(2\pi)^2 \delta(p_z')\, e^{-(p_x'^2 + p_y'^2)/k_c^2} \tag{10}$$

is the Fourier transform of $\langle V\rangle$ and $f(\omega)$ is the Fermi function.

In order to make further analytic progress we assume that $qv_F > \Delta^2/k_c V_F$. It is then possible to integrate over the magnitude of $\mathbf{p}$ and the polar angle, with the result that

$$\alpha_L = \frac{m^2}{v_s^2 \rho_{\text{ion}}}\left(\frac{p_F^2}{3m}\right)^2 \int \frac{\mathrm{d}\phi}{2\pi}\int \frac{\mathrm{d}\omega}{2\pi}$$
$$\times [f(\omega) - f(\omega+\omega_s)] I(\omega, \theta) \tag{11}$$

where

$$I(\omega, \theta) = (2Re\, K(z_0) - 1)(2Re\, K(z_0^+) - 1)$$
$$-2(\Delta/k_c v_F \sin\theta)^2$$
$$\times Re\left[K(z_0)K(z_0^+) \times \frac{i\sqrt{\pi}w(z_0^+) - i\sqrt{\pi}w(z_0)}{z_0^+ - z_0}\right.$$
$$\left. + K(z_0^+)K^*(z_0)\frac{i\sqrt{\pi}w(z_0^+) - i\sqrt{\pi}w(z_0^*)}{z_0^+ - z_0^*}\right] \tag{12}$$

z_0 is the solution of the equation

$$z_0 = \frac{2(\omega + i/2\tau)}{k_c v_F \sin\theta} + i\sqrt{\pi}(\Delta/k_c V_F \sin\theta)^2\, W(z_0)$$

$$z_0^+ = z_0(\omega+\omega_s)$$

$$K(z_0) = [1 - i\sqrt{\pi}(\Delta/k_c v_F \sin\theta)^2 W(z_0)]^{-1}$$

$$\sin\theta = (1-\sin^2\phi\sin^2\alpha)^{1/2}$$

and α is the angle between $\mathbf{q}$ and $\mathbf{B}$. Note that $[2Re\, K(z_0)-1] = N(\omega,\theta)$ is the angular dependent density of states found by Brandt *et al.*[2]. In deriving equation (12) we have made use of the fact that when $ql \gg 1$ the only electrons contributing to the absorption are those moving essentially perpendicular to the direction of propagation of the wave.

From the form of equation (12) it can be immediately seen that under the conditions specified above the attenuation will be strongly anisotropic. In particular we note that if $\sin\theta = 0$, the density of states is singular and the function

$$I(\omega, \theta=0) = \tfrac{1}{2}[1+(\omega^2-\Delta^2)/|\omega^2-\Delta^2|] \quad (13)$$

the BCS coherence factor. In general the form of equation (12) is such that for an arbitrary geometry the attenuation coefficient can only be obtained by numerical computation. It is possible, however, to make further progress in the simple case of parallel propagation ($\mathbf{q}\|\mathbf{B}$). If we assume $\omega_s\tau \ll 1$, the usual experimental situation, then keeping terms to first order in ω_s we find

$$\alpha_L{}^s = -\frac{m^2}{v_s{}^2\rho_{\text{ion}}}\left(\frac{p_F{}^2}{3m}\right)^2 \omega_s \int \frac{d\omega}{2\pi}\frac{\partial f(\omega)}{\partial\omega} \times\Bigg[[2Re\, K(z_0)-1]^2 - 2(\Delta/k_c v_F)^2 \times Re\Bigg[K^2(z_0)\, i\sqrt{\pi}W'(z_0) + |K(z_0)|^2 \times \frac{i\sqrt{\pi}W(z_0) - i\sqrt{\pi}W(z_0^*)}{z_0 - z_0^*}\Bigg]\Bigg]. \quad (14)$$

In the region of validity of the theory $(H_{c_2}-B) \leqq H_{c_2}$ the parameter $(\Delta/k_c v_F) \cong (\Delta/\Delta_{BCS}) \ll 1$ and therefore $K(z_0)$ can be expanded in powers of $(\Delta/k_c v_F)^2$. Further if we use the property of the $W(z)$ function

$$i\sqrt{\pi}W(z^*) = [i\sqrt{\pi}W(z)]^* \quad (15)$$

we find

$$\alpha_L{}^s = -\frac{m^2}{v_s{}^2\rho_{\text{ion}}}\left(\frac{p_F{}^2}{3m}\right)^2 \omega_s \int \frac{d\omega}{2\pi}\frac{\partial f(\omega)}{\partial\omega} \times\Bigg[1 - 2(\Delta/k_c v_F)^2 \times\Bigg[\frac{k_c l\, Im\,(i\sqrt{\pi}W(z_0))}{1+(\Delta/k_c v_F)^2 k_c l\, Im\,(i\sqrt{\pi}W(z_0))} - Re\, i\sqrt{\pi}W'(z_0)\Bigg]\Bigg]. \quad (16)$$

2. RESULTS

The attenuation coefficient for longitudinal sound propagating parallel to the magnetic field in Nb at 4·2°K has been obtained from equation (16), by numerical computation, for three values of $k_c l$. The results of this calculation are shown in Fig. 1, where $\alpha_r{}^2 = (\alpha_s - \alpha_n)^2/(\alpha_n)^2$ is plotted vs. B/H_{c_2}. The physical parameters used in determining these curves were; Δ^2, which was taken from Eilenberger's paper[3], H_{c_2} (4·2°K) = 2680 G[4], $V_F = 3{\cdot}0\times10^7$ cm/sec and the density of states[5] $N(0) = 5{\cdot}6\times10^{34}$ erg/cm³. We would like to point out the following features of Fig. 1: (1) $\alpha_r{}^2$ varies markedly with mean free path even when $k_c l \gg 1$ and follows the trend found experimentally by Forgan and Gough[6], this is in contrast to the theoretical predictions of Maki[7], also shown in this figure, which are mean free path independent under these conditions; (2) $\alpha_r{}^2$ is, apart from a small region close to H_{c_2}, a linear function of $(H_{c_2}-B)$ down to fields $B/H_{c_2} = 0{\cdot}98$ at which point it starts to deviate from straight line behavior this is also consistent with Ref. [6]. It is not possible at this time to make an absolute comparison of theory with experiment as the only theoretical results we have are at a temperature of 4·2°K, whereas the available experimental

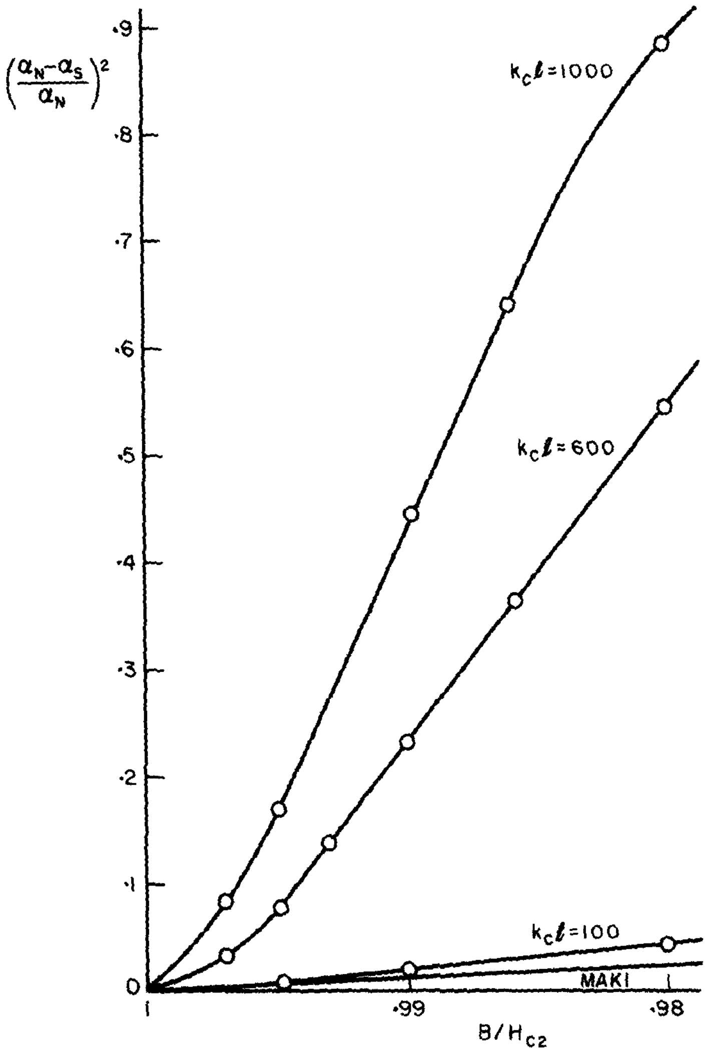

Fig. 1. The normalized attenuation in the mixed state at 4·2°K as a function of B/H_{c_2}.

data[6, 8] is for temperatures of 2·02°K and 1·3°K. We do not expect, however, the absolute magnitude of the attenuation to change much in this temperature range, it would therefore appear that the theoretical values of α_r^2 agree qualitatively with the experiment results of Forgan and Gough whose purest sample gives $\alpha_r^2 \cong 0{\cdot}4$ at $B/H_{c_2} \cong 0{\cdot}98$. A detailed discussion of these points together with calculations of the attenuation as a function of direction of propagation for different temperatures will be given elsewhere.

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