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Localization: The Effect of a Weak Magnetic Field

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The effect of a weak magnetic field on the diffusion of noninteracting electrons in a disordered system is studied in a nonlinear σ-model context. The effective Lagrangian describing the soft modes of the system in the weak field limit is derived. The result does not have the simple form that has been suggested by several authors. Therefore the crossover of the system under a weak perturbing magnetic field is not analogous to that found in spin systems.

KEY WORDS: Localization; field theory; magnetic field; effective Lagrangian; crossover.

1. INTRODUCTION

In 1979 Wegner[1] showed how the diffusive behavior of noninteracting electrons in a disordered system can be mapped onto a nonlinear σ model. The effective Lagrangian is expressed in terms of matrix fields Q varying in an ensemble which depends on the symmetries of the original electronic system. This leads to a formal derivation of the one-parameter scaling theory of Abrahams *et al.*[2]; however, recently Kravtsov and Lerner[3] have pointed out that there may be some instabilities in this theory. In this paper we will discuss the effect of a uniform magnetic field on a d-dimensional disordered electron system in the extreme weak field limit for $d > 2$, using Wegner's formalism. We find that even if we put aside the possible problems pointed out by Kravtsov and Lerner, the form of the symmetry-breaking term in the effective Lagrangian is not of the simple type conjectured in the literature, and hence the calculation of the crossover from the zero-field fixed point is more complicated than has been supposed.

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Following on from Wegner's original work, several authors wrote down a nonlinear σ model describing noninteracting electrons in a disordered system in the presence of a magnetic field **B**. This model had a unitary symmetry as opposed to the orthogonal symmetry found in the zero-field case. Hikami[4] obtained the zero-field model by mapping diagrams found in perturbation theory onto those found in the nonlinear σ model. He then argued that for $\mathbf{B} \neq \mathbf{0}$ the diffusion pole disappears in the particle–particle channel leading to the lowering of the symmetry from orthogonal to unitary. Efetov *et al.*[5] started from a model Hamiltonian with various interactions and derived nonlinear σ models with various corresponding symmetries; in particular, they found that the presence of a magnetic field gave the unitary case. Later Pruisken[6] elaborated on this work. He found that in two dimensions for strong fields an extra term of a topological nature is important and used this to discuss the quantum Hall effect.

It is now natural to ask how the crossover from orthogonal to unitary symmetry occurs. It has been suggested[4,7] that a relevant analogy may be with anisotropic spin models. The crossover phenomena associated with such systems have been thoroughly studied in the context of nonlinear σ models.[8,9] If it is assumed that the symmetry-breaking term that occurs when the field is switched on is of this type (i.e., appears as a single mass term in the field theory), then the calculation of the crossover has been performed by Wegner.[10] However, it is the purpose of this paper to show that the symmetry breaking is not of this type.

We should point out at this stage that several papers in the literature have addressed the weak field problem and crossover. Although they do not use field-theoretic techniques, their assumption about the symmetry breaking is of the kind discussed above. Khmel'nitskii and Larkin[7] use the spin analogy, Belitz[11] has a massive propagator for the particle–particle modes, and Oppermann[12] has a phenomenological interpolation between the two models that leads to the same crossover exponent calculated by Wegner. All this contrasts with a comment by Levine *et al.*[13] to the effect that the nature of this crossover is still open. We now go on to a systematic investigation of this question.

In Section 2 we begin from a model Hamiltonian for noninteracting electrons with disorder in the presence of a magnetic field and obtain a field-theoretic representation of the problem. The weak field limit is discussed in Section 3 and an effective Lagrangian obtained. The form of the symmetry-breaking term is found in Section 4 and we conclude by making some general comments in Section 5.

2. FIELD-THEORETIC FORMULATION

In this section we begin from a simple model of noninteracting electrons in a magnetic field **B** and a random potential $V(x)$. We derive a field-theoretic representation for the Green's functions of the system, which allows the effects of the disorder to be studied. The approach and the notation will closely follow McKane and Stone.[14]

The starting point is the model Hamiltonian

$$H = -(\nabla - ie\mathbf{A})^2 + V(x) = H_0 + V(x) \tag{2.1}$$

where $\hbar = c = 1$, $m = \frac{1}{2}$. As usual, the random potential is taken to have a Gaussian distribution

$$P[V] = \exp -\frac{1}{2\gamma} \int d^d x \, V^2(x) \tag{2.2}$$

In the zero-field case[14] a functional integral identity for the Green's functions involving n real replica fields, with $n \to 0$, was used in order to perform the average over the random potential. In this case, where H is no longer symmetric, but is Hermitian, the corresponding identity involves n complex replica fields ϕ^α ($\alpha = 1, 2, ..., n$):

$$G(x, y; V, E \pm i\eta) = \pm i \lim_{n \to 0} \int D\phi \, D\phi^* \, \phi^1(x) \, \phi^{1*}(y)$$
$$\times \exp - \int d^d z \, [\pm i\phi^{\alpha *}(H - E)\phi^\alpha + \eta\phi^{\alpha *}\phi^\alpha] \tag{2.3}$$

In this and subsequent equations repeated indices are summed over. Actually, there are a number of other representations we might have used. For instance, we could have worked with two-component, real, commuting fields instead of complex commuting fields or with anticommuting (Grassmann) fields or even with a combination of anticommuting and commuting fields. However, we find the complex commuting representation (2.3) the clearest.

The quantities we want to study are those that may show critical behavior[14]:

$$\overline{G^e(x, y; V, E + i\eta) \, G^e(x, y; V, E - i\eta)} \tag{2.4a}$$

$$\overline{G^e(x, y; V, E + i\eta) \, G^h(x, y; V, E - i\eta)} \tag{2.4b}$$

$$\overline{G^h(x, y; V, E + i\eta) \, G^e(x, y; V, E - i\eta)} \tag{2.4c}$$

$$\overline{G^h(x, y; V, E + i\eta) \, G^h(x, y; V, E - i\eta)} \tag{2.4d}$$

where the average is over the random potential V,

$$G^{\mathrm{e}}(x, y; V, E \pm i\eta) = is \lim_{n \to 0} \int D\boldsymbol{\phi}_{s,\mathrm{e}}\, D\boldsymbol{\phi}^*_{s,\mathrm{e}}\, \phi^1_{s,\mathrm{e}}(x)\, \phi^{1*}_{s,\mathrm{e}}(y) \times \exp - \int d^dz\, \phi^{\alpha*}_{s,\mathrm{e}}[is(H-E)+\eta]\phi^{\alpha}_{s,\mathrm{e}} \tag{2.5a}$$

$$G^{\mathrm{h}}(x, y; V, E \pm i\eta) = is \lim_{n \to 0} \int D\boldsymbol{\phi}_{s,\mathrm{h}}\, D\boldsymbol{\phi}^*_{s,\mathrm{h}}\, \phi^1_{s,\mathrm{h}}(x)\, \phi^{1*}_{s,\mathrm{h}}(y) \times \exp - \int d^dz\, \phi^{\alpha*}_{s,\mathrm{h}}[is(H^T-E)+\eta]\phi^{\alpha}_{s,\mathrm{h}} \tag{2.5b}$$

and where $s = \pm$ for $E \pm i\eta$ and the symbol e (h) stands for electron (hole). Although the expectation is that the particle–particle channels [Eqs. (2.4a) and (2.4d)] will become noncritical when the magnetic field is switched on, in order to study the crossover, we need to investigate all four combinations. We note for further use that

$$G^{\mathrm{e}}(x, y; V, E \pm i\eta) = G^{\mathrm{h}}(y, x; V, E \pm i\eta) \tag{2.6a}$$

$$G^{\mathrm{e}}(x, y; V, E \pm i\eta) = G^{\mathrm{h}*}(x, y; V, E \mp i\eta) \tag{2.6b}$$

For reasons that will become apparent, we use (2.6a) to rewrite the expressions (2.4) in the form $G(x, y; V, E+i\eta)\, G(y, x; V, E-i\eta)$. Now using Eq. (2.5), we can write the four expressions (2.4) as

$$\langle \phi^1_{+,\mathrm{e}}(x)\, \phi^{1*}_{-,\mathrm{h}}(x)\, \phi^1_{-,\mathrm{h}}(y)\, \phi^{1*}_{+,\mathrm{e}}(y) \rangle \tag{2.7a}$$

$$\langle \phi^1_{+,\mathrm{e}}(x)\, \phi^{1*}_{-,\mathrm{e}}(x)\, \phi^1_{-,\mathrm{e}}(y)\, \phi^{1*}_{+,\mathrm{e}}(y) \rangle \tag{2.7b}$$

$$\langle \phi^1_{+,\mathrm{h}}(x)\, \phi^{1*}_{-,\mathrm{h}}(x)\, \phi^1_{-,\mathrm{h}}(y)\, \phi^{1*}_{+,\mathrm{h}}(y) \rangle \tag{2.7c}$$

$$\langle \phi^1_{+,\mathrm{h}}(x)\, \phi^{1*}_{-,\mathrm{e}}(x)\, \phi^1_{-,\mathrm{e}}(y)\, \phi^{1*}_{+,\mathrm{h}}(y) \rangle \tag{2.7d}$$

where the brackets $\langle \cdots \rangle$ are defined by

$$\langle \sigma(x, y) \rangle = \frac{\int DV\, o(x, y; V) \exp - (1/2\gamma) \int d^dz\, V^2(z)}{\int DV \exp - (1/2\gamma) \int d^dz\, V^2(z)} \tag{2.8}$$

and where $o(x, y; V)$ is given by

$$\begin{aligned} o(x, y; V) &= \lim_{n \to 0} \int D\boldsymbol{\phi}_{+,\mathrm{e}}\, D\boldsymbol{\phi}^*_{+,\mathrm{e}}\, D\boldsymbol{\phi}_{+,\mathrm{h}}\, D\boldsymbol{\phi}^*_{+,\mathrm{h}}\, D\boldsymbol{\phi}_{-,\mathrm{e}}\, D\boldsymbol{\phi}^*_{-,\mathrm{e}}\, D\boldsymbol{\phi}_{-,\mathrm{h}}\, D\boldsymbol{\phi}^*_{-,\mathrm{h}}\, \sigma(x, y) \\ &\quad \times \exp - \int d^dz\, \{ i\phi^{\alpha*}_{+,\mathrm{e}}[H - E - i\eta]\phi^{\alpha}_{+,\mathrm{e}} + i\phi^{\alpha*}_{+,\mathrm{h}}[H^T - E - i\eta]\phi^{\alpha}_{+,\mathrm{h}} \\ &\quad - i\phi^{\alpha*}_{-,\mathrm{e}}[H - E + i\eta]\phi^{\alpha}_{-,\mathrm{e}} - i\phi^{\alpha*}_{-,\mathrm{h}}[H^T - E + i\eta]\phi^{\alpha}_{-,\mathrm{h}} \} \end{aligned} \tag{2.9}$$

So far we have attempted to write down explicit formulas for the sake of clarity. However, it is obvious that they have also become very cumbersome. To make subsequent expressions more compact, we introduce the following notation:

(i) An index $a=(\alpha, i)$, where $i=1, 2$ labels electrons and holes, so that $a=1, 2,\ldots, 2n$.

(ii) An index $A=(a, s)$, where $s=1, 2$ labels the plus and minus sectors, so that $A=1, 2,\ldots, 4n$.

As in the zero-field case,[14] we now define composite fields $Q^{AB}(x)$, which essentially play the role of the combination $\phi^A(x)\,\phi^{B*}(x)$. More specifically, they are introduced by eliminating the fourth-order term in the exponent of Eq. (2.9) (which is generated by performing the average over V) using

$$\int DQ \exp -\int d^dz\, \{\tfrac{1}{2}\operatorname{tr} Q^2 + i\gamma^{1/2}\phi^{A*}Q^{AB}C^{BC}\phi^C\}$$

$$=\int DQ \exp -\int d^dz\, \{\tfrac{1}{2}\operatorname{tr} Q^2\} \exp -\int d^dz\, \{\tfrac{1}{2}\gamma[\phi_+^{a*}\phi_+^a - \phi_-^{a*}\phi_-^a]^2\} \quad (2.10)$$

The minus sign in the fourth-order term means that it is invariant under the group of transformations $U(2n, 2n)$. Consequently, the matrix Q in (2.10) is pseudo-Hermitian:

$$(Q_{++}^{ab})^* = Q_{++}^{ba}, \qquad (Q_{+-}^{ab})^* = -Q_{-+}^{ba}, \qquad (Q_{--}^{ab})^* = Q_{--}^{ba} \quad (2.11)$$

It is useful when dealing with the noncompact symmetry groups that arise in this problem to define the matrix

$$C=\begin{bmatrix} I_{2n} & 0 \\ 0 & -I_{2n} \end{bmatrix} \quad (2.12)$$

Then the condition (2.11) becomes $CQ^\dagger C = Q$.

Substituting Eq. (2.10) into Eq. (2.9) after averaging, we can now perform the ϕ integrals, since they are Gaussian. In fact, it is easier not to deal with specific operators $\sigma(x, y)$, but to introduce source terms for the ϕ fields; functional differentiation then gives any desired operator. As in Ref. 14, we can show that the same quantities can be obtained by introducing source terms for the Q fields and functionally differentiating with respect to these new sources. This leads us to the generating functional

$$Z[J] = \lim_{n\to 0} \int \left(\prod_{A=1}^{4n} D\phi^A\, D\phi^{A*}\right) DQ$$

$$\times \exp -\int d^dz\, (\tfrac{1}{2}\operatorname{tr} Q^2 + i\phi^{A*}A^{AB}C^{BC}\phi^C - J^{AB}Q^{AB}) \quad (2.13)$$

where

$$A^{AB}(Q)=\gamma^{1/2}Q^{AB}+\delta^{\alpha\beta}\begin{bmatrix} H_0-E-i\eta & 0 & 0 & 0 \\ 0 & H_0^T-E-i\eta & 0 & 0 \\ 0 & 0 & H_0-E+i\eta & 0 \\ 0 & 0 & 0 & H_0^T-E+i\eta \end{bmatrix} \tag{2.14}$$

Performing the ϕ integration in Eq. (2.13), we obtain

$$Z[J]=\lim_{n\to 0}\int DQ\exp-\left\{\mathrm{Tr}\ln\ A(Q)+\int d^dz\,[\tfrac{1}{2}\,\mathrm{tr}\,Q^2-J^{AB}Q^{AB}]\right\} \tag{2.15}$$

where here Tr means trace over the spatial as well as the internal indices. Before proceeding further with the determination of the effective Lagrangian, let us write down the form of the expressions (2.4), and thus (2.7), in terms of the Q fields. They are, respectively,

$$-\gamma^{-1}\langle Q^{11}_{+-,\mathrm{eh}}(x)\,Q^{11}_{-+,\mathrm{he}}(y)\rangle \tag{2.16a}$$

$$-\gamma^{-1}\langle Q^{11}_{+-,\mathrm{ee}}(x)\,Q^{11}_{-+,\mathrm{ee}}(y)\rangle \tag{2.16b}$$

$$-\gamma^{-1}\langle Q^{11}_{+-,\mathrm{hh}}(x)\,Q^{11}_{-+,\mathrm{hh}}(y)\rangle \tag{2.16c}$$

$$-\gamma^{-1}\langle Q^{11}_{+-,\mathrm{he}}(x)\,Q^{11}_{-+,\mathrm{eh}}(y)\rangle \tag{2.16d}$$

The new Lagrangian has terms linear in Q, which we deal with in the usual way, that is, we introduce $\hat{Q}=Q-\langle Q\rangle$ and expand in $\hat{Q}$. The vertices for the $\hat{Q}$ fields can be found via the identity

$$\begin{aligned}\mathrm{Tr}\ln A&=\mathrm{Tr}\ln(T+\hat{T})=\ln\det(T+\hat{T})\\&=\ln\det T+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\underbrace{\mathrm{Tr}(T^{-1}\hat{T}T^{-1}\hat{T}\cdots T^{-1}\hat{T})}_{n\text{ times}}\end{aligned} \tag{2.17}$$

where $T=A(\langle Q\rangle)$ and $\hat{T}^{AB}(x)=\gamma^{1/2}\hat{Q}^{AB}(x)$. We then determine $\langle Q\rangle$ by asking that

$$\frac{\delta}{\delta Q^{AB}(x)}\left\{\mathrm{Tr}\ln A(Q)+\frac{1}{2}\int d^dz\,\mathrm{tr}\,Q^2\right\}\Bigg|_{\hat{Q}=0}=0 \tag{2.18}$$

which gives

$$\langle Q^{AA}\rangle=-\gamma^{-1/2}T^{-1AA}_{xx} \tag{2.19}$$

where in this case repeated indices do not imply summation. All off-diagonal terms $\langle Q^{AB}\rangle$, $A \neq B$, are zero.

The solution to (2.19) is replica-independent and therefore this equation can be written simply as four separate equations:

$$\langle Q_{\pm\pm,\mathrm{ee}}\rangle = -\gamma^{1/2} G_0^{\mathrm{e}}(x, x; \pm) \tag{2.20a}$$

$$\langle Q_{\pm\pm,\mathrm{hh}}\rangle = -\gamma^{1/2} G_0^{\mathrm{h}}(x, x; \pm) \tag{2.20b}$$

where

$$G_0^{\mathrm{e}}(x, y; \pm) = [H_0 - E \mp i\eta + \gamma^{1/2}\langle Q_{\pm\pm,\mathrm{ee}}\rangle]_{xy}^{-1} \tag{2.21a}$$

$$G_0^{\mathrm{h}}(x, y; \pm) = [H_0^T - E \mp i\eta + \gamma^{1/2}\langle Q_{\pm\pm,\mathrm{hh}}\rangle]_{xy}^{-1} \tag{2.21b}$$

The subscript zero on the Green's functions (2.21) indicates that they are zeroth-order approximations to Eqs. (2.5) (after averaging), formulated in a simple self-consistent scheme.

Having eliminated the linear term, we now go on to calculate the quadratic terms in $\hat{Q}$ in the Lagrangian. For our purposes it is sufficient to write down only the terms involving $\hat{Q}_{+-}$ and $\hat{Q}_{-+}$. These are

$$\begin{aligned} -\frac{1}{2}\int d^dx\, d^dy\, \{&\hat{Q}^{\dagger\alpha\beta}_{-+,1}(x)\, C^{\mathrm{ee}}_{+-}(x, y)\, \hat{Q}^{\beta\alpha}_{+-,1}(y) \\ &+ \hat{Q}^{\dagger\alpha\beta}_{-+,2}(x)\, C^{\mathrm{eh}}_{+-}(x, y)\, \hat{Q}^{\beta\alpha}_{+-,2}(y) \\ &+ \hat{Q}^{\dagger\alpha\beta}_{-+,3}(x)\, C^{\mathrm{he}}_{+-}(x, y)\, \hat{Q}^{\beta\alpha}_{+-,3}(y) \\ &+ \hat{Q}^{\dagger\alpha\beta}_{-+,4}(x)\, C^{\mathrm{hh}}_{+-}(x, y)\, \hat{Q}^{\beta\alpha}_{+-,4}(y) \\ &+ \hat{Q}^{\dagger\alpha\beta}_{+-,1}(x)\, C^{\mathrm{ee}}_{-+}(x, y)\, \hat{Q}^{\beta\alpha}_{-+,1}(y) \\ &+ \hat{Q}^{\dagger\alpha\beta}_{+-,2}(x)\, C^{\mathrm{eh}}_{-+}(x, y)\, \hat{Q}^{\beta\alpha}_{-+,2}(y) \\ &+ \hat{Q}^{\dagger\alpha\beta}_{+-,3}(x)\, C^{\mathrm{he}}_{-+}(x, y)\, \hat{Q}^{\beta\alpha}_{-+,3}(y) \\ &+ \hat{Q}^{\dagger\alpha\beta}_{+-,4}(x)\, C^{hh}_{-+}(x, y)\, \hat{Q}^{\beta\alpha}_{-+,4}(y)\} \end{aligned} \tag{2.22}$$

where for notational convenience we have substituted 1 for the combination (eh), 2 for (ee), 3 for (hh), and 4 for (he), in the Q fields. For instance, $Q^{\alpha\beta}_{\pm\mp,\mathrm{eh}}(x) = Q^{\alpha\beta}_{\pm\mp,1}(x)$. We have also used the symmetries exhibited in Eq. (2.11):

$$(Q^\dagger)^{\alpha\beta}_{\pm\mp,1} = -Q^{\alpha\beta}_{\pm\mp,4}, \qquad (Q^\dagger)^{\alpha\beta}_{\pm\mp,2} = -Q^{\alpha\beta}_{\pm\mp,2}, \qquad (Q^\dagger)^{\alpha\beta}_{\pm\mp,3} = -Q^{\alpha\beta}_{\pm\mp,3} \tag{2.23}$$

The definitions of the C factors are

$$C^{\text{ee}}_{\pm\mp}(x, y) = \delta^d(x-y) - \gamma G^{\text{e}}_0(x, y; \pm)\, G^{\text{e}}_0(x, y; \mp) \tag{2.24a}$$

$$C^{\text{eh}}_{\pm\mp}(x, y) = \delta^d(x-y) - \gamma G^{\text{e}}_0(x, y; \pm)\, G^{\text{h}}_0(x, y; \mp) \tag{2.24b}$$

$$C^{\text{he}}_{\pm\mp}(x, y) = \delta^d(x-y) - \gamma G^{\text{h}}_0(x, y; \pm)\, G^{\text{e}}_0(x, y; \mp) \tag{2.24c}$$

$$C^{\text{hh}}_{\pm\mp}(x, y) = \delta^d(x-y) - \gamma G^{\text{h}}_0(x, y; \pm)\, G^{\text{h}}_0(x, y; \mp) \tag{2.24d}$$

We have not included the massive $\hat{Q}_{++}$ and $\hat{Q}_{--}$ fields in Eq. (2.22); these only manifest themselves in the final form of the effective Lagrangian via a constraint, in a way that is by now familiar.[1,14–16] The $\hat{Q}_{\pm\mp,1}$ and $\hat{Q}_{\pm\mp,4}$ fields may also become massive due to the presence of the magnetic field. However, we do not eliminate them via a constraint—they are included because it is precisely this possible mass generation for small fields that we wish to study.

Before looking at the exact form of effective Lagrangian, we need to investigate the structure of the C factors in (2.24a)–(2.24d) in some detail. This we now go on to do.

3. FORM OF THE EFFECTIVE LAGRANGIAN

In this section we present a calculation of the effective Lagrangian in the presence of a magnetic field.

We note that the degeneracy of the C factors [defined in Eq. (2.24)] for the particle–particle and particle–hole channels has been lifted due to the presence of the magnetic field. General considerations allow the electron and hole Green's functions to be written as[17]

$$G^{\text{e}}_0(x, y; \mathbf{A}) = \exp\left(ie \int_y^x \mathbf{A} \cdot d\mathbf{s}\right) \tilde{G}^{\text{e}}_0(x-y; \mathbf{B}) \tag{3.1a}$$

$$G^{\text{h}}_0(x, y; \mathbf{A}) = \exp\left(-ie \int_y^x \mathbf{A} \cdot d\mathbf{s}\right) \tilde{G}^{\text{h}}_0(x-y; \mathbf{B}) \tag{3.1b}$$

where $\tilde{G}^{\text{e,h}}_0$ depend only on the magnetic field $\mathbf{B}$ and are translationally invariant. Introducing these results into Eq. (2.24), it is clear that the particle–hole propagators C^{eh} and C^{he} are translationally invariant, as in the zero-field case. However, in the particle–particle propagators C^{ee} and C^{hh}, the gauge factors do not cancel. It is the lack of translational invariance in these latter terms that makes the magnetic field problem more difficult than the zero-field one—a simple small-momentum expansion is not possible. Let us now discuss the corresponding expansion required to obtain an effective Lagrangian suitable for investigating critical behavior in the weak field limit.

In weak magnetic fields, if the cyclotron radius is much larger than the electron mean free path l, we may use the quasiclassical approximation[18]

$$G_0^e(x, y; \mathbf{A}) = \exp\left(ie\int_y^x \mathbf{A}\cdot d\mathbf{s}\right)\tilde{G}_0(x-y; \mathbf{B}=\mathbf{0}) \tag{3.2a}$$

$$G_0^h(x, y; \mathbf{A}) = \exp\left(-ie\int_y^x \mathbf{A}\cdot d\mathbf{s}\right)\tilde{G}_0(x-y; \mathbf{B}=\mathbf{0}) \tag{3.2b}$$

where the integral is along a straight line connecting x and y. Notice that $\tilde{G}_0$ is the same for electrons and holes.

To evaluate the C factors, we substitute Eq. (3.2) into Eq. (2.24). Let us consider C^{eh} and C^{he} first, since the treatment of these is exactly as in the zero-field case.[14] In terms of $\tilde{G}_0$ they are

$$C^{he}(p) = C^{eh}(p) = \int d^dz\, e^{ipz}[\delta^d(z) - \gamma\tilde{G}_0(z;+)\,\tilde{G}_0(z;-)] \tag{3.3}$$

We are interested in critical behavior and so we expand for small momentum

$$C^{he}(p) = C^{eh}(p) = C_0 + C_2 p^2 + \cdots \tag{3.4}$$

We want to identify C_0 and C_2 with physical quantities; it will turn out that they depend on the zeroth-order density of states and diffusion constant in zero field. To see this, recall that from Eqs. (2.4b), (2.4c), (2.16b), (2.16c), and (2.23)

$$\overline{|G_0^e(x, y; +)|^2} = \gamma^{-1}\langle Q^{\dagger 11}_{+-,2}(x)\, Q^{11}_{-+,2}(y)\rangle_0 \tag{3.5a}$$

$$\overline{|G_0^h(x, y; +)|^2} = \gamma^{-1}\langle Q^{\dagger 11}_{+-,3}(x)\, Q^{11}_{-+,3}(y)\rangle_0 \tag{3.5b}$$

Using Eq. (3.2) on the left-hand side and evaluating the right-hand side using Eq. (2.22), we obtain

$$\overline{|\tilde{G}_0(x-y; \mathbf{B}=\mathbf{0}, +)|^2} = \gamma^{-1}[C^{eh}(x, y)]^{-1} = \gamma^{-1}[C^{he}(x, y)]^{-1} \tag{3.6}$$

The zeroth-order density of states $\bar{\rho}_0$ is given by [Eq. (3.13) of Ref. 14]

$$\pi\bar{\rho}_0/\eta \underset{\eta\to 0}{\sim} \overline{|\tilde{G}_0(p; +)|^2}|_{p^2=0} = \gamma^{-1}C^{-1}(p)|_{p^2=0} \tag{3.7}$$

using Eq. (3.6). Therefore

$$C_0 \sim \eta/\pi\gamma\bar{\rho}_0 \tag{3.8}$$

The zeroth-order diffusion constant $\bar{D}_0$ is given by [Eq. (3.14) of Ref. 14]

$$\frac{\pi\bar{D}_0\bar{\rho}_0}{2\eta^2} \underset{\eta\to 0}{\sim} -\frac{\partial}{\partial p^2}\overline{|\tilde{G}_0(p;+)|^2}|_{p^2=0} \tag{3.9}$$

But from Eq. (3.6)

$$\frac{\partial}{\partial p^2}\overline{|\tilde{G}_0(p;+)|^2} = -\frac{\gamma^{-1}C_2}{C_0^2} \tag{3.10}$$

Therefore

$$C_2 \sim \bar{D}_0/2\pi\gamma\bar{\rho}_0 \tag{3.11}$$

Equation (3.8) shows the existence of a particle in the Q_{+-} particle–hole channel whose mass tends to zero as $\eta\to 0$, as we would expect.

Now consider an electron–electron term in (2.22), which, dropping all indices, can be written as

$$-\frac{1}{2}\int d^dx\, d^dy\, Q^\dagger(x)\left\{\delta^d(x-y) - \gamma\left[\exp\left(2ie\int_y^x \mathbf{A}\cdot d\mathbf{s}\right)\right]|\tilde{G}_0(x-y;+|^2\right\}Q(y) \tag{3.12}$$

At low fields we can take the gauge term out of the curly brackets and write expression (3.12) as

$$-\frac{1}{2}\int d^dx\, d^dy\, Q^\dagger(x)\left\{\delta^d(x-y) - \gamma|\tilde{G}_0(x-y;+)|^2\right\}\left[\exp\left(2ie\int_y^x \mathbf{A}\cdot d\mathbf{s}\right)\right]Q(y) \tag{3.13}$$

Note that the factor in the curly brackets is just $C^{\mathrm{eh}}(x,y)=C^{\mathrm{he}}(x,y)$, so that apart from the gauge terms, effectively only one type of C factor appears. Therefore from now on we drop the e and h superscripts on C and refer simply to $C(x-y)$, by which we will mean $C^{\mathrm{eh}}(x,y)$ and $C^{\mathrm{he}}(x,y)$.

We now use the fact that for any functions $\mathbf{A}$ and Q and a straight path[19]

$$\exp 2ie\int_y^x \mathbf{A}\cdot d\mathbf{s}\, Q(y) = \{\exp[(y-x)(\nabla - 2ie\mathbf{A})]\}\, Q(x) \tag{3.14}$$

to write Eq. (3.13) as

$$-\frac{1}{2}\int d^dx\, d^dy\, Q^\dagger(x)\, C(x-y)\exp\{(y-x)[\nabla_x - 2ie\mathbf{A}(x)]\}\, Q(x) \tag{3.15}$$

If we assume the exponential to be small, which is valid for the case $l(2eB)^{1/2} \ll 1$, then keeping only second-order terms as in Eq. (3.4), we find

$$-\frac{1}{2}\int d^dx\, d^dy\, Q^\dagger(x)\, C(x-y)\left\{1+\frac{1}{2}(y-x)^2[\nabla_x - 2ie\mathbf{A}(x)]^2 + \cdots\right\} Q(x) \tag{3.16}$$

Introducing a new variable $z = x - y$, and integrating over z gives

$$-\frac{1}{2}\int d^dx\, Q^\dagger(x)[C_0 - C_2(\nabla - 2ie\mathbf{A})^2]\, Q(x) \tag{3.17}$$

where C_0 and C_2 are as before. Similar considerations apply to hole–hole terms with e changing to $-e$ in Eq. (3.17).

Having identified the general features that appear, we can now approximate the Lagrangian (2.22), keeping only those parts likely to be relevant to critical behavior. As already discussed, we work to lowest order in a momentum expansion; in addition, we ignore all loops of the massive Q_{++} and Q_{--} excitations. This is exactly as in the zero-field case[14]—the $Q_{+-,1}$ and $Q_{+-,4}$ excitations [which from Eq. (3.17) might be thought to become massive] are not eliminated in this way, since it is precisely this aspect of the symmetry-breaking that we are trying to study. Thus, the effective Lagrangian is

$$\begin{aligned}\frac{1}{2}C_2\int d^dx\, \{&\hat{Q}^{\alpha\beta}_{ij,4}[-(\nabla - 2ie\mathbf{A})^2]\,\hat{Q}^{\beta\alpha}_{ji,1}\\ &+\hat{Q}^{\alpha\beta}_{ij,2}[-\nabla^2]\,\hat{Q}^{\beta\alpha}_{ji,2}\\ &+\hat{Q}^{\alpha\beta}_{ij,3}[-\nabla^2]\,Q^{\beta\alpha}_{ji,3}+\hat{Q}^{\alpha\beta}_{ij,1}[-(\nabla + 2ie\mathbf{A})^2]\,\hat{Q}^{\beta\alpha}_{ji,4}\}\end{aligned} \tag{3.18}$$

where $i, j = 1, 2$ and where the overall factor of C_2 is there to agree with the quadratic Q_{+-} and Q_{-+} terms. The Q_{++} and Q_{--} terms are eliminated using a constraint in the usual way.[14] Since we are interested in the weak-field limit, the construction is as in the zero-field case, with the effect of a field included as a perturbation. When $\mathbf{B} = \mathbf{0}$, Eq. (3.18) can be written as

$$\frac{1}{2}C_2\int d^dx\,\{\mathrm{tr}\,\hat{Q}[-\nabla^2]\hat{Q}\} \tag{3.19}$$

where

$$\hat{Q} = \begin{bmatrix}\hat{Q}_2 & \hat{Q}_1\\ \hat{Q}_4 & \hat{Q}_3\end{bmatrix} \tag{3.20}$$

and where the trace is over all indices. The field $\hat{Q}$ is complex; to make contact with the zero-field formulation, we decompose it into real and imaginary parts. This displays the underlying orthogonal symmetry. The constraint that determines the $\hat{Q}_{++}$ and $\hat{Q}_{--}$ fields is found by asking that the real and imaginary parts of $\hat{Q}$ are obtained from $\langle Q \rangle$ by a (pseudo)-orthogonal transformation

$$\hat{Q}(x) = O^{\times}(x) \begin{bmatrix} \mathrm{Re}\langle Q_{++} \rangle & 0 \\ 0 & \mathrm{Re}\langle Q_{--} \rangle \end{bmatrix} O(x) + iO^{\times}(x) \begin{bmatrix} \mathrm{Im}\langle Q_{++} \rangle & 0 \\ 0 & \mathrm{Im}\langle Q_{--} \rangle \end{bmatrix} O(x) \tag{3.21}$$

where $O^{\times}$ is defined using the C matrix introduced in Eq. (3.12):

$$O^{\times} = CO^{T}C \tag{3.22}$$

and $O^{\times}O = I_{4n}$. But $\mathrm{Re}\langle Q_{++} \rangle = \mathrm{Re}\langle Q_{--} \rangle$ and $\mathrm{Im}\langle Q_{++} \rangle = -\mathrm{Im}\langle Q_{--} \rangle = -\pi\gamma^{1/2}\bar{\rho}_0$. Therefore

$$\hat{Q}(x) = \text{const} - i\pi\gamma^{1/2}\bar{\rho}_0 O^{\times}(x)\, CO(x) \tag{3.23}$$

The constant term is irrelevant, since we can shift $\hat{Q}$ by the constant in Eq.(3.19). Therefore $\hat{Q}(x)$ obeys

$$\hat{Q}^2(x) = -\pi^2\gamma(\bar{\rho}_0)^2 I_{4n} \tag{3.24}$$

This is the $O(2n, 2n)/O(2n) \times O(2n)$ nonlinear σ model. There has been a doubling up of the number of fields as compared with the usual $\mathbf{B} = \mathbf{0}$ treatment, where the $O(n, n)/O(n) \times O(n)$ model is found. However, these two models give identical results in the $n = 0$ limit.

When the magnetic field is switched on, the orthogonal symmetry is broken, as can be seen from Eq. (3.18). Our interest in this paper is in the form of the propagator when $\mathbf{B} \neq \mathbf{0}$. We therefore now discuss the structure of the free part of the effective Lagrangian coming from the Q_{+-} and Q_{-+} fields, and leave aside the nature of the interactions generated by the constraint.

4. STRUCTURE OF THE PROPAGATOR

From Eq. (3.18) the free part of the effective Lagrangian for the $\mathbf{B} \neq \mathbf{0}$ problem is

$$\mathscr{L}^{\text{eff}}_{\text{free}} = \frac{1}{2} \int d^d x\, \{\hat{Q}^{\alpha\beta}_{+-,4}[-(\nabla - 2ie\mathbf{A})^2]\, \hat{Q}^{\beta\alpha}_{-+,1} + \hat{Q}^{\alpha\beta}_{+-,2}[-\nabla^2]\, \hat{Q}^{\beta\alpha}_{-+,2} + \hat{Q}^{\alpha\beta}_{+-,3}[-\nabla^2]\, \hat{Q}^{\beta\alpha}_{-+,3} + \hat{Q}^{\alpha\beta}_{+-,1}[-(\nabla + 2ie\mathbf{A})^2]\, \hat{Q}^{\beta\alpha}_{-+,4}\} \tag{4.1}$$

The $\hat{Q}_{-+}$ sector gives an identical contribution to the $\hat{Q}_{+-}$ sector and they have been amalgamated in Eq. (4.1). Also, the $\hat{Q}$ fields have been scaled by $C_2^{1/2}$, so that the free part is of a conventional form, with the coupling multiplying the interactions.[14] From Eq. (4.1) the propagators for the various $\hat{Q}$ fields can be found. Specifically, we have to diagonalize the quadratic forms. This cannot be done in general and so we use the gauge invariance of (4.1) to make a particular gauge choice. It seems easiest to use the Landau gauge, which in d dimensions takes the form $\mathbf{A} = (-Bx_2, 0, 0,..., 0)$, so that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ has only $F_{12} = -F_{21} = B$ different from zero.

In the case of the $\hat{Q}_2$ and $\hat{Q}_3$ fields we obviously go to a momentum space representation and find propagators $1/p^2$, showing the massless modes for these particle–hole channels. The $\hat{Q}_1$ and $\hat{Q}_4$ terms are what really interest us; we will only consider the electron–electron term from now on; clearly, changing e to $-e$ gives the hole–hole propagator. Therefore we wish to diagonalize

$$\int d^dx\, \hat{Q}^\dagger(x)(\nabla - 2ie\mathbf{A})^2\, \hat{Q}(x)$$

$$= \int d^dx\, \hat{Q}^\dagger(x) \left(\left\{ \frac{\partial}{\partial x_1} + 2ieBx_2 \right\}^2 + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \right) \hat{Q}(x) \qquad (4.2)$$

The diagonalization of Eq. (4.2) is familiar from the solution of the Schrödinger equation for a particle moving in a magnetic field.[20] One finds that (4.2) equals

$$\sum_{n,p} \hat{Q}^\dagger_{n,p} \lambda_{n,p} \hat{Q}_{n,p} \qquad (4.3)$$

where

$$\hat{Q}(x) = \sum_{n,p} \chi_{n,p}(x)\, \hat{Q}_{n,p} \qquad (4.4)$$

and

$$\lambda_{n,p} = (2n+1)\alpha^2 + p_3^2 + \cdots + p_d^2 \qquad (4.5)$$

Here $\alpha^2 = 2eB$ and the sum over the momenta p is understood to be an integral over $p_1, p_3,..., p_d$. The χ functions equal

$$\chi_{n,p}(x) = \{\exp[i(p_1x_1 + p_3x_3 + \cdots + p_dx_d)]\} \times \left(\frac{\alpha}{2^n n!\, \pi^{1/2}} \right)^{1/2} H_n(\alpha\xi) \exp\frac{-\alpha^2\xi^2}{2} \qquad (4.6)$$

where $\xi = x_2 - \alpha^2 p_1$ and H_n is a Hermite polynomial. One could now go ahead and perform calculations in this basis, but it has been suggested in the literature that for weak fields, at least, on diagonalization in momentum space the result would be $p^2 + m^2$, where m^2 is a mass depending on the magnetic field. In order to check whether this is so, let us write (4.2) as

$$\int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} \hat{Q}^\dagger(p)\, F(p, p')\, \hat{Q}(p') \tag{4.7}$$

where

$$F(p, p') = -\int d^d x\, [\exp(ip \cdot x)(\nabla - 2ie\mathbf{A})^2[\exp(-ip' \cdot x)] \tag{4.8}$$

Expanding the exponential factors as in Eq. (4.4), we obtain

$$F(p, p') = 2\pi^{1/2}\alpha\delta(p_1 - p'_1)\,\delta(p_3 - p'_3)\cdots\delta(p_d - p'_d)$$
$$\times \exp\frac{ip_1(p_2 - p'_2)}{\alpha^2} \exp -\frac{p_2^2 + p_2'^2}{2\alpha^2} \sum_{n=0}^{\infty} \frac{2n+1}{n!\,2^n} H_n\left(\frac{p_1}{\alpha}\right) H_n\left(\frac{p'_1}{\alpha}\right) \tag{4.9}$$

The claim is that for small α^2,

$$F(p, p') \sim \delta^d(p - p')[p^2 + m^2] \tag{4.10}$$

Evaluating the sum in (4.9), we find that

$$F(p, p') = \delta^d(p - p')\left(p^2 - 2i\alpha^2 p_1 \frac{\partial}{\partial p_2} - \alpha^4 \frac{\partial^2}{\partial p_2^2}\right) \tag{4.11}$$

For small α^2 this is not of the form (4.10), that is, the symmetry-breaking term is not a simple mass term. This can be seen directly from Eq. (4.8). What we have shown in this section is that, no matter how small α^2 is, we have to use the representation (4.4) for calculational purposes.

5. CONCLUSION

The purpose of this paper has been to clarify various statements in the literature concerning the form of the crossover from the zero-field fixed point describing a noninteracting electron moving in a random potential when a magnetic field is turned on. A systematic study was carried out in the context of field theory and an effective Lagrangian describing the system in a weak magnetic field was derived in what is by now a familiar fashion. We found that the structure of the propagators was such that even for infinitesimally small magnetic fields a purely momentum space representation could not be used. This shows that this particular crossover problem is considerably more complicated than has been supposed by many authors. In particular, we have shown that the symmetry breaking is not of a simple mass type, as found in spin systems formulated as nonlinear σ models. Calculations using the basis described in Section 4 have been carried out in certain instances where only the $n=0$ mode was important: in the strong field case[21,22] and in a problem where a finite mass difference between the $n=0$ and higher modes implied that this mode determined the critical behavior.[23] In the symmetry-breaking problem we have been discussing in this paper, the situation is more complicated, and it is probable that all modes have to be retained in any calculation of the crossover exponent.

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