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LONGITUDINAL ULTRASONIC ATTENUATION IN CLEAN TYPE II SUPERCONDUCTORS*

Hilda A. Cerdeira, W.N. Cottingham[†] and A. Houghton

Department of Physics, Brown University, Providence, R.I.

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The attenuation of longitudinal sound in clean type II superconductors, near H_{c2}, has been calculated using a non-perturbative method. It is shown that at low frequencies the attenuation rate depends strongly on the direction of propagation of the wave, and on impurity concentration.

IN THIS paper we present a procedure for calculating the transport coefficients of clean type II superconductors near H_{c2} which does not depend on an expansion in powers of the order parameter. The theory is used to calculate the attenuation coefficient of longitudinal sound; which is shown to depend strongly on the direction of propagation of the sound wave.

In the low frequency limit $\omega_s < \pi T_{co}$, provided $ql >> 1$, the attenuation coefficient of longitudinal sound can be simply expressed in terms of the density–density correlation function

$$\alpha = \mathrm{Re} \frac{q^2}{i\,\omega_s\,\rho_{ion} v_s} \left(\frac{P_F^2}{3m}\right) <[n,n]>(\mathbf{q},\omega_s) \qquad (1)$$

In (1) $\mathbf{q}$ and ω_s are the wave vector and frequency of the sound wave, p_F is the Fermi momentum and $<[n,n]>(\mathbf{q},\omega_s)$ is the volume average of the density–density correlation function. The correlation function is obtained by analytic continuation of the thermal product $<[n,n]>(\mathbf{q},\omega_0)$ from the set of discrete points $\omega_0 = 2m\pi T$ to $z = \omega_s - i\delta$. If we assume that, in the vicinity of H_{c2}, the nearly uniform and constant magnetic field inside the superconductor can be replaced by its space average, the magnetic induction $\mathbf{B}$, then the thermal product can be written

$$<[n,n]>(\mathbf{q},\omega_0) = 2T \sum_{\omega} \int d^3r \int d^3r'\, e^{-i\mathbf{q}.(r-r')} \Big[G_{\omega+}(\mathbf{r},\mathbf{r}')\, G_{\omega}(\mathbf{r}',\mathbf{r}) - \int d^3r_1 \int d^3r_2\, G^0_{-\omega+}(\mathbf{r}-\mathbf{r}_1)\, G_{\omega+}(\mathbf{r}',\mathbf{r}_1)\, V(\mathbf{r}_1,\mathbf{r}_2)\, G_{\omega}(\mathbf{r}_2,\mathbf{r}')\, G^0_{-\omega}(\mathbf{r}_2-\mathbf{r})\Big] \qquad (2)$$

The Green's function $G_\omega(\mathbf{r},\mathbf{r}')$ appearing in (2) satisfies the equation

$$G_\omega(\mathbf{r},\mathbf{r}') = G^0_\omega(\mathbf{r}-\mathbf{r}') - \int d^3r_1 \int d^3r_2\, G^0_\omega(\mathbf{r}-\mathbf{r}_1)\, V(\mathbf{r}_1,\mathbf{r}_2)\, G^0_{-\omega}(\mathbf{r}_1-\mathbf{r}_2)\, G_\omega(\mathbf{r}_2,\mathbf{r}'), \qquad (3)$$

here $G^0(\mathbf{r}-\mathbf{r}')$ is the normal metal Green's function in the absence of a magnetic field, the function V is defined by:

$$V(\mathbf{r}_1,\mathbf{r}_2) = \Delta(\mathbf{r}_1)\,\Delta^*(\mathbf{r}_2)\, e^{-ieB(x_1+x_2)(y_1-y_2)},$$

$$\omega = (2n+1)\pi T \quad \text{and} \quad \omega_+ = \omega + \omega_0. \qquad (4)$$

It has been shown[1] that iterative solutions to (3) lead in certain cases to unphysical results even in the limit $B \to H_{c2}$; recently, however, Brandt *et al.*[2] have solved (3) approximately by a method which avoids the iteration procedure.

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† Permanent Address: Physics Department, University of Bristol, Bristol, England.

These authors note that when $(H_{c2} - B)\, H_{c2}$ the order parameter $\Delta(\mathbf{r})$ is given by the Abrikosov solution of the Ginzburg–Landau equations

$$\Delta(\mathbf{r}) = \sum_n C_n\, e^{inay}\, e^{-eB[x - (an/2eB)]^2} \quad (5)$$

and therefore $G_\omega(\mathbf{r},\mathbf{r}')$ when considered as a function of sum and difference coordinates, has the periodicity of the flux line lattice with respect to the sum coordinate. Fourier analyzing (3) and noting that the dominant contribution to G_ω is obtained if V is replaced by $<V>$, an average over its sum coordinate, they obtain in the infinite mean free path limit

$$G_\omega(\mathbf{p},\mathbf{k}) = \delta_{k,0}\left[i\omega - \xi_p + \frac{i\sqrt{\pi}\ \Delta^2}{k_c v_F \sin\theta} W\left(\frac{i\omega + \xi_p}{k_c v_F \sin\theta}\right)\right]^{-1} \quad (6)$$

where the wave vector $k_c = (2eB)^{1/2}$, is inversely proportional to the spacing between flux lines, $\Delta^2 = |\Delta|^2$, $\xi_p = P^2/2m - \mu$, θ is the angle between $\mathbf{p}$ and $\mathbf{B}$ and

$$W(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} dt\, \frac{e^{-t^2}}{z - t} \quad (7)$$

Using this result $<[n,n]>(\mathbf{q},\omega_0)$ is given by

$$<[n,n]>(\mathbf{q},\omega_0) = 2T\sum_\omega \int \frac{d^3p}{(2\pi)^3}[G_{\omega-}(\mathbf{p}+\mathbf{q},0)\, G_\omega(\mathbf{p},0) - G_{\omega+}(\mathbf{p}+\mathbf{q},0) M_\omega(\mathbf{p},\mathbf{q})\, G_\omega(\mathbf{p},0)] \quad (8)$$

where

$$M_\omega(\mathbf{p},\mathbf{q}) = \int \frac{d^3p'}{(2\pi)^3} G^0_{-\omega+}(\mathbf{p}+\mathbf{q}-\mathbf{p}')\, V(\mathbf{p}',0)\, G^0_{-\omega}(\mathbf{p}-\mathbf{p}') \quad (9)$$

and

$$V(\mathbf{p}',0) = (\Delta^2/k_c^2)\, 2(2\pi)^2\, \delta(p_z')\, e^{-(p_x'^2 + p_y'^2)/k_c^2} \quad (10)$$

In this paper we restrict our calculations to the clean limit k_ce >> 1, in this case scattering effects due to impurities are included in the theory by simply replacing $\tilde{\omega}$ by $\omega = \omega(1 + i/2\tau|\omega|)$. In principle the order parameter and the electromagnetic vertices should also be renormalized, however, as k_ce >> 1 and qe >> 1 these renormalization effects are negligible to order $1/k_c$e and $1/q$e respectively. We should point out that as $k_c \to 0$ when $T \to T_c$ this theory is not valid too close to the transition temperature. After making this replacement and performing the analytic continuation of equation (7) we obtain

$$\alpha_L = -\frac{q}{v_s \rho_{ion}}\left(\frac{p_F^2}{3m}\right)^2 \mathrm{Im}\int \frac{d\omega}{2\pi i}[f(\omega+\omega_s) - f(\omega)] \times [<[n,n]>(\mathbf{q};\, \omega+\omega_s + i/2\tau,\, \omega + i/2\tau) - <[n,n]>(\mathbf{q};\, \omega+\omega_s + i/2\tau, \omega - i/2\tau)] \quad (11)$$

where

$$<[n,n]>(\mathbf{q};\omega+\omega_s + i/2\tau,\, \omega - i/2\tau) = 2\int\frac{d^3p}{(2\pi)^3} G_{\omega+\omega_s+i/2\tau}(\mathbf{p}+\mathbf{q},0)\, G_{\omega-i/2\tau}(\mathbf{p},0) \times \left[1 - \int\frac{d^3p}{(2\pi)^3} G^0_{-(\omega+\omega_s+i/2\tau)}(\mathbf{p}+\mathbf{q}-\mathbf{p}')\, G^0_{-(\omega-i/2\tau)}(\mathbf{p}+\mathbf{p}')\, V(\mathbf{p}',0)\right] \quad (12)$$

and $f(\omega)$ is the Fermi function.

In order to make further analytic progress we assume that $qv_F > \Delta$. It is then possible to integrate over the magnitude of $\mathbf{p}$ and the polar angle, with the result that

$$\alpha_L = \frac{m^2}{v_s^2 \rho_{ion}}\left(\frac{p_F^2}{3m}\right)^2 \int\frac{d\phi}{2\pi}\int\frac{d\omega}{2\pi}[f(\omega) - f(\omega+\omega_s)]\, I(\omega,\theta) \quad (13)$$

where

$$I(\omega,\theta) = \Big\{(2\,\mathrm{Re}\,k(z_0) - 1)(2\,\mathrm{Re}\,k(z_0^+) - 1) - 2(\Delta/k_c v_F \sin\theta)^2 \times \mathrm{Re}\Big[k(z_0)k(z_0^+)\frac{i\sqrt{\pi}\,\omega(z_0^+) - i\sqrt{\pi}\,\omega(z_0)}{z_0^+ - z_0} + k(z_0^+)\, k^*(z_0)\, \frac{i\sqrt{\pi}\omega(z_0^+) - i\sqrt{\pi}\omega(z_0^*)}{z_0^+ - z_0^*}\Big]\Big\}, \quad (14)$$

$$z_0 = \frac{2(\omega + i/2\tau)}{k_c v_F \sin\theta} + i\sqrt{\pi}\,(\Delta/k_c v_F \sin\theta)^2 \omega(z_0)$$

$$z_0^+ = z_0(\omega + \omega_s)$$

$$k(z_0) = [1 - i\sqrt{\pi}(\Delta/k_c v_F \sin\theta)^2 \omega'(z_0)]^{-1}$$

$$\sin\theta = (1 - \sin^2\phi\, \sin^2\alpha)^{1/2}$$

and α is the angle between $\mathbf{q}$ and $\mathbf{B}$. Note that $[2\,\mathrm{Re}\,k(z_0) - 1] = N(\omega,\theta)$ is the angular dependent density of states found by Brandt *et al.* In deriving (12) we have made use of the fact that when $ql >> 1$

the only electrons contributing to the absorption are those moving essentially perpendicular to the direction of propagation of the wave.

The form of equation (14) is such that for an arbitrary geometry the attenuation coefficient can only be obtained by numerical computation. It is possible, however to make further progress in the simple case of parallel propagation (**q** ∥ **B**). If we consider first the case of most practical interest $\omega_s\tau \ll 1$; keeping terms to first order in ω_s we get

$$\alpha_L^s = \frac{m^2}{v_s^2\rho_{ion}}\left(\frac{P_F^2}{3m}\right)^2 \omega_s \int \frac{d\omega}{\partial\omega}\frac{\partial f(\omega)}{\partial\omega}\Big\{(2\,\mathrm{Re}\,k(z_0)-1)^2 - 2(\Delta/k_c v_F)^2\,\mathrm{Re}\Big[k^2(z_0)\, i\sqrt{\pi}W'(z_0) + |k(z_0)|^2\frac{i\sqrt{\pi}W(z_0)-i\sqrt{\pi}W(z_0^*)}{2i\,\mathrm{Im}\,z_0}\Big]\Big\} \quad (15)$$

In the region of validity of the theory ($H_{c2} - B) \ll H_{c2}$) the parameter $(\Delta/k_c v_F) \cong \Delta/\Delta_{BCS} \ll 1$, therefore, $K(z_0)$ and $W'(z_0)$ can be expanded in powers of $(\Delta/k_c v_F)^2$. Further, using the property of the $W(z)$ function

$$i\sqrt{\pi}\,W(z^*) = [i\sqrt{\pi}W(z)]^* \quad (16)$$

and the definition of z_0 we see that

$$(\Delta/k_c v_F)^2\,\frac{i\sqrt{\pi}W(z_0) - i\sqrt{\pi}W(z_0^*)}{2i\,\mathrm{Im}\,z_0} = \frac{(\Delta/k_c v_F)\,k_c e\;\mathrm{Im}\,(i\sqrt{\pi}W(z_c))}{[1+(\Delta/k_c v_F)^2 k_c e\;\mathrm{Im}(i\sqrt{\pi}W(z_0))]} \quad (17)$$

Using this result and the expansion of $K(z_0)$ and $W'(z_0)$ the attenuation coefficient to leading order in $(\Delta/k_c v_F)$ reduces to

$$\alpha_L^s = -\frac{m^2}{v_s^2\rho_{ion}}\left(\frac{P_F^2}{3m}\right)^2\frac{\omega_s}{2}\Big\{1 - 2(\Delta/k_c v_F)^2\Big[\frac{k_c e\,\mathrm{Im}\,(i\sqrt{\pi}W(i/k_c e)}{1+(\Delta/k_c v_F)^2 k_c e\,\mathrm{Im}\,(i\sqrt{\pi}W(i/k_c e))} - \mathrm{Re}\,(i\sqrt{\pi}W''(i/k_c e))\Big]\Big\}. \quad (18)$$

From the form of equation (18) it is evident that the attenuation coefficient in this geometry can only be represented by a simple power series in $(\Delta/k_c v_F)$ if $(\Delta/k_c v_F)^2\, k_c e \ll 1$; in which case

$$\alpha_L^s = -\frac{m^2}{v_s^2\,\rho_{ion}}\left(\frac{P_F^2}{3m}\right)^2\frac{\omega_s}{2}\,[1 - 2\sqrt{\pi}(\Delta/k_c v_F)^2 k_c e + 0(\Delta^4)]. \quad (19)$$

If on the other hand $\omega_s\tau \gg 1$ the attenuation coefficient can only be expanded in powers of Δ^2 if $(\Delta^2/k_c v_F\omega_s) \ll 1$. The result of the expansion in this case is

$$\alpha_s = \alpha_N + 0(\Delta^4) \quad (20)$$

which agrees with the earlier calculation of Cyrot and Maki.

It is more interesting to examine the form of the attenuation for perpendicular propagation (**q** ⊥ **B**). As before, the quasi particles which contribute to the attenuation move in the plane perpendicular to **q**. Contained in this group are quasi particles with momentum **p** parallel or antiparallel to **B**; in this direction, $\sin\theta = 0$, the density of states is singular and the function

$$I(\omega,\theta = 0) = \tfrac{1}{2}[1 + (\omega^2 - \Delta^2)/|\omega^2 - \Delta^2|] \quad (21)$$

the B.C.S. coherence factor. It is therefore clear that the relative attenuation $(\alpha_s - \alpha_N)/\alpha_N$ will contain terms proportional to Δ in addition to terms proportional to Δ^2.

In this geometry the exact form of the attenuation can only be found by numerical calculation. The detailed results of a calculation of the attenuation as a function of direction of propagation, magnetic field, and impurity concentration will be presented elsewhere. Finally it should be emphasized that the theory presented in this note is only valid provided $\mathbf{q}e \gg 1$ and $k_c e \gg 1$; to the best of our knowledge experiments have not yet been performed such that both of these criteria are satisfied.

REFERENCES

1. See for example CYROT M. and MAKI K., *Phys. Rev.* **196**, 433 (1967); MAKI K., *Phys. Rev.* **156**, (1967).
2. BRANDT U., PESCH W. and TEWORDT L., *Z. Phys.* **201**, 209 (1967).

Nous avons calculé l'atténuation d'une onde de son longitudinale dans un supraconducteur propre de type II, au voisinage de H_{c2}, a l'aide d'une methode non-perturbative. On montre qu 'aux basses frequency le taux d'atténuation dépend fortement de la direction de propagation de l'onde, et de la concentration des impuretés.